Paired-sample z-test

1. Assumptions

- **Experimental Design:** The sample forms two treatment groups, where each subject in one group is paired with a subject in the other group.
- **Null Hypothesis:** The average of the pair differences is not significantly different than zero.
- **Population Distribution:** Arbitrary.
- **Sample Size:** The sample size of each treatment group is greater than or equal to 30.

2. Inputs for the paired-sample z-test

- Sample size: n
- Sample mean: $\bar{d}$ of the differences, where $d_i = x_{1i} - x_{2i}$, $x_{1i} \in$ Treatment Group 1; $x_{2i} \in$ Treatment Group 2.
- Sample standard deviation: s_d (SD of the differences)
- Standard error of mean: $\mathrm{SE}_{\text{mean}} = \dfrac{s_d}{\sqrt{n}}$
- Null hypothesis value: 0
- The level of the test: α

3. Five Steps for Performing the Test of Hypothesis

1. State null and alternative hypotheses:
$$H_0 : \mu = 0, \quad H_1 : \mu \neq 0$$
2. Compute test statistic:
$$z = \frac{\bar{d} - \mu}{\mathrm{SE}_{\text{mean}}},$$
assuming the null hypothesis value 0 for μ.

3. Compute $100(1-\alpha)\%$ confidence interval I for z.
4. If $z \in I$, accept H_0; if $z \notin I$, reject H_1 and accept H_0.
5. Compute p-value.

4. Discussion

For the difference variable $d_i = x_{1i} - x_{2i}$,, the central limit theorem insures that $\bar{d} \sim N(\mu_x, \sigma_x^2/\sqrt{n})$. The paired two-sample z-test reduces to a one-sample z-test on the differences d_i.

4. A Sample Problem

Freedman, Pisani, and Purves, p. p. 476: A legislative committee wants to see if there is a significance difference in tax revenue between the proposed new tax law and the existing tax law. The committee has a staff member randomly choose 100 representative tax returns. For the ith return, it computes the tax x_{2i} using the proposed new tax law, and compares it to the tax x_{1i} paid under the existing law. The staff member then computes the differences $d_i = x_{2i} - x_{1i}$ and tests whether there is a significant difference between the proposed new law and the existing law.
Here are the summary statistics:

$$n = 100 \qquad \bar{d} = -219 \qquad s_d = 725 \qquad c = 0 \qquad \alpha = 0.05$$

$$\mathrm{SE}_{\text{mean}} = \frac{s_x}{\sqrt{n}} = \frac{725}{\sqrt{100}} = 72.5.$$

The five steps of the z-test:

1. State the null and alternative hypotheses:

$$H_0 = 0, \qquad H_1 \neq 0$$

2. Compute the test statistic:

$$z = \frac{\bar{x} - \mu}{\mathrm{SE}_{\text{mean}}} = \frac{-219 - 0}{72.5} = -3.02$$

3. Compute a $100(1-\alpha)\%$ confidence interval I. $z \sim N(0,1)$, so $I = [-1.96, 1.96]$.

4. Determine whether to accept or reject H_0: $-3.02 \notin [-1.96, 1.96]$, so reject H_0.

5. Compute the p-value: if u is standard normal,

$$P(z \leq -u) = P(u \leq -3.02) = 0.0013.$$

By the symmetry of the normal curve,

$$P(z \leq u) = P(3.02 \leq u) = 0.0013.$$

Thus $p = 0.0013 + 0.0013 = 0.0026$.

As in the one-sample z-test, small p-values computed with the standard normal table can be very different than the p-values computed with the t-tables, even when $n > 30$.