One-sample *t*-test

1. Assumptions

- Experimental Design: The sample forms a single treatment group.
- Null Hypothesis: The population mean of the treatment group is not significantly different from a hypothesized constant *c*.
- Population Distribution: Approximately normal.
- Sample Size: Arbitrary.

2. Inputs for the t-test

- Sample size: n
- Sample mean: \bar{x}
- Sample standard deviation: s_x
- Standard error of mean: $SE_{mean} = \frac{s_x}{\sqrt{n}}$
- Null hypothesis value: c
- The level of the test: α

3. Five Steps for Performing the Test of Hypothesis

1. State null and alternative hypotheses:

$$H_0: \mu = c, \quad H_1: \mu \neq c$$

2. Compute test statistic:

$$t = \frac{\bar{x} - \mu}{\mathrm{SE}_{\mathrm{mean}}}$$

assuming the null hypothesis value for μ .

- 3. Compute $100(1-\alpha)\%$ confidence interval *I* for *t*, using the *t*-table with n-1 degrees of freedom.
- 4. If $t \in I$, accept H_0 ; if $t \notin I$, reject H_1 and accept H_0 .
- 5. Compute *p*-value with statistical software.

4. Discussion

If n < 30, the sample standard deviation s_x may be quite variable and not necessarily close to the population standard deviation σ_x . This means that $SE_z = (\bar{x} - \mu)/s_x$ is not necessarily normal, because of the random variable s_x in the denominator. However, $(n-1)(s_x^2/\sigma_x^2)$ can be shown to have a χ^2 distribution with n-1 degrees of freedom. It can also be shown that \bar{x} and s_x^2 are independent. This gives us

$$t = \frac{\bar{x} - \mu}{\mathrm{SE}_{\mathrm{mean}}} = \frac{\bar{x} - \mu}{\sqrt{\frac{s_x^2}{n}}} = \frac{\frac{\bar{x} - \mu}{\sqrt{n}}}{\sqrt{\frac{(n-1)s_x^2}{(n-1)\sigma_x^2}}} = \frac{Z}{\sqrt{\frac{U}{n-1}}}$$

where

$$Z = \frac{\bar{x} - \mu}{\sigma_x^2}$$
 and, $U = \frac{(n-1)s_x^2}{\sigma_x^2}$

Z and U are also independent. By definition a random variable t has a t distribution with n-1 degrees of freedom if it is of the form

$$\frac{Z}{\sqrt{\frac{U}{n-1}}},$$

where $Z \sim N(0, 1)$, $U \sim \chi^2(n-1)$, and Z and U are independent. Therefore, we can use the t distribution to find confidence intervals for the t statistic.

4. A Sample Problem

The carbon monoxide (CO) level in a manufacturing plant is supposed to be about 50 parts per million (ppm). However the actual CO levels are quite variable. Five CO measurements are taken at various times during the day:

58 63 48 52 68

Test the null hypotheses that $\mu = 50$ ppm for the CO concentration at the manufacturing plant.

Solution: We have these inputs:

$$n = 5$$
 $\bar{x} = 57.8$ $s_x = 8.07$ $c = 50$ $\alpha = 0.05$
 $SE_{mean} = \frac{s_x}{\sqrt{n}} = \frac{8.07}{\sqrt{5}} = 3.61.$

The five steps of the t-test:

1. State the null and alternative hypotheses:

$$H_0 = 50, \qquad H_1 \neq 50$$

2. Compute the test statistic:

$$t = \frac{\bar{x} - \mu}{\mathrm{SE}_{\mathrm{mean}}} = \frac{57.8 - 50.0}{3.61} = 2.16$$

- 3. Compute a $100(1 \alpha)\%$ confidence interval *I*: use the *t*-table with n 1 = 5 1 = 4 degrees of freedom to show that [-2.776, 2.776] is a 95% confidence interval for *t*.
- 4. Determine whether to accept or reject H_0 : 2.16 $\in [-2.776, 2.776]$, so accept H_0 .
- 5. Let SAS or R compute the p-value: 0.0969.