The Cauchy-Schwartz Inequality and $|r| \leq 1$

Statement of Theorem

For any u and v in the vector space V,

$$(u \cdot v)^2 \leq ||u||^2 \, ||v||^2, \tag{1}$$

where $u \cdot v$ is the inner product of u and v; and $||u||$ is the norm of u. Taking square roots of both sides of (1) gives another form of the Cauchy-Schwartz inequality:

$$|u \cdot v| \leq ||u|| \, ||v||. \tag{2}$$

Review of Inner Product and Norm

An **inner product** of a vector space V, also called a dot product, satisfies the following properties. Assume that u and v are vectors in V and that s and t are real numbers. We have

1. $s(u + v) = su + sv$
2. $(s + t)u = su + tu$
3. $s(tu) = (st)u$

The **norm** of a vector v is defined by $||v|| = \sqrt{u \cdot u}$. It satisfies these properties. Assume that u and v are vectors and that s is a real number.

1. $||u + v|| \leq ||u|| + ||v||$ (Triangle Inequality)
2. $||su|| = |s| \cdot ||u||$
3. $||u|| = 0 \iff u = 0$

Proof of Theorem

If u and v are collinear, say $v = tu$, we have equality in (1):

$$(u \cdot v)^2 = [u \cdot (tu)]^2 = t^2(u \cdot u)^2 = t^2||u||^4 = ||u||^2||tu||^2 = ||u||^2||v||^2 \quad (3)$$

If u and v are not collinear, then $tu + v \neq 0$ for all real values of t. This means that $||tu + v|| > 0$ for all $t \in R$. Since $u \cdot u = ||u||^2$, we have

$$\begin{aligned} ||tu + v||^2 &> 0 \\ (tu + v) \cdot (tu + v) &> 0 \\ (tu) \cdot (tu) + (tu) \cdot v + v \cdot (tu) + ||v||^2 &> 0 \\ ||u||^2 t^2 + 2(u \cdot v)t + ||v||^2 &> 0. \end{aligned}$$

Setting

$$a = ||u||^2, \quad b = 2(u \cdot v), \quad c = ||v||^2, \quad (4)$$

the quadratic inequality $at^2 + bt + c > 0$, is true for all values of t. Therefore the quadratic equation

$$at^2 + bt + c = 0 \quad (5)$$

has no real solutions. Because the solutions of this quadratic equation are given by

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

(5) has no real solutions if and only if the discriminant is negative: $b^2 - 4ac < 0$. Using the values in (4),

$$\begin{aligned} [2(u \cdot v)]^2 - 4||u||^2||v||^2 &< 0 \\ 4(u \cdot v)^2 - 4||u||^2||v||^2 &< 0 \\ (u \cdot v)^2 - ||u||^2||v||^2 &< 0, \end{aligned}$$

so

$$(u \cdot v)^2 < ||u||^2||v||^2. \quad (6)$$

Combining the cases in (3) and (6) gives the complete Cauchy-Schwartz inequality:

$$(u \cdot v)^2 \leq ||u||^2||v||^2.$$

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r is between -1 and 1

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two data vectors in R^n. Define $\overline{x}$ and $\overline{y}$ to be the two mean vectors

$$\overline{x} = \left(\sum_{i=1}^{n} x_i, \ldots, \sum_{i=1}^{n} x_i \right)$$

and

$$\overline{y} = \left(\sum_{i=1}^{n} y_i, \ldots, \sum_{i=1}^{n} y_i \right)$$

($\overline{x}$ and $\overline{y}$ are also in R^n.) Then

$$r = \frac{s_{xy}}{s_x s_y} = \frac{\dfrac{\sum_{i=1}^{n}(x_i - \overline{x})(x_i - \overline{x})}{n-1}}{\sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}\sqrt{\dfrac{\sum_{i=1}^{n}(y_i - \overline{y})^2}{n-1}}}$$

$$= \frac{\dfrac{(x - \overline{x}) \cdot (y - \overline{y})}{n-1}}{\sqrt{\dfrac{||x - \overline{x}||^2}{n-1}}\sqrt{\dfrac{||y - \overline{y}||^2}{n-1}}} = \frac{\dfrac{(x - \overline{x}) \cdot (y - \overline{y})}{n-1}}{\dfrac{||x - \overline{x}|| \, ||y - \overline{y}||}{n-1}} = \frac{(x - \overline{x}) \cdot (y - \overline{y})}{||x - \overline{x}|| \, ||y - \overline{y}||} \qquad (7)$$

From (2), we have $|u \cdot v|/||u|| \, ||v|| \leq 1$, so (7) ≤ 1 and r is between -1 and 1.