On Geometric Spanners of Euclidean Graphs and their Applications in Wireless Networks

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***Abstract*— We consider the problem of constructing a bounded-degree planar geometric spanner for a unit disk graph modeling a wireless network. The related problem of constructing a planar geometric spanner of a Euclidean graph has been extensively studied in the literature. It is well known that the Delaunay subgraph is a planar geometric spanner with stretch factor $C_{del} \approx 2.42$; however, its degree may not be bounded. Significant work has been done on developing algorithms for constructing bounded-degree planar geometric spanners of Euclidean graphs. Our first result presents a very simple linear time algorithm for constructing a subgraph of the Delaunay graph with stretch factor $\rho = 1 + 2\pi(k \cos \frac{\pi}{k})^{-1}$ and degree bounded by k, for any integer parameter $k \geq 14$. This result immediately implies an algorithm for constructing a planar geometric spanner of a Euclidean graph with stretch factor $\rho \cdot C_{del}$ and degree bounded by k, for any integer parameter $k \geq 14$. Our second contribution lies in developing the structural results necessary to transfer our analysis and algorithm from Euclidean graphs to unit disk graphs. We obtain a very simple strictly-localized algorithm that, given a unit disk graph embedded in the plane, constructs a planar geometric spanner with the above stretch factor and degree bound. The two results dramatically improve the previous results in all aspects, as shown in the paper.**

I. Introduction

Topology control of wireless ad-hoc networks deployed in the plane is a fundamental problem in the area of wireless computing [23], [25], [26]. Topology control is used, for example, to construct a topology of the original network that is amenable to routing or other networking applications. Some desirable properties of the resulting topology include (1) planarity: the underlying graph should be planar to allow, for example, guaranteed and efficient routing such as geometric routing ([5], [16]); (2) bounded stretch factor: for any two devices in the network there should be a path connecting them in the topology whose length is close to the length of the shortest path connecting the pair in the original network; (3) bounded degree: each device maintains links to only a constant number of devices in its communication range, thus minimizing interference and saving energy; and (4) constructible locally: the construction of the network topology should be distributed, simple, and strictly-localized in the sense that each point constructs and maintains its links in the topology based only on the information from neighboring devices without exchanging or propagating global information (see [7] for a formal definition). The problem of computing efficient topologies for ad-hoc wireless networks has been extensively considered in the literature [14], [18], [21], [23], [24], [25], [26], [28], [29], [32], [33], [34].

Since wireless networks are usually modeled as unit disk graphs in the Euclidean plane, the corresponding problem becomes to construct geometric spanners for unit disk graphs. The related problem of constructing geometric spanners of Euclidean graphs (i.e., the complete graphs on a given set of points in the plane) is a very important problem that has received a lot of attention in the literature due to its numerous applications in the fields of computational geometry, wireless computing, and computer graphics (for example, see the recent book [22] for a survey on geometric spanners and their applications in networks). Dobkin et al. [11] showed that the Delaunay graph is a planar geometric spanner of the Euclidean graph with stretch factor $(1 + \sqrt{5})\pi/2 \approx 5.08$. This ratio was further improved by Keil et al [15] to $C_{del} = 2\pi/(3\cos(\pi/6)) \approx 2.42$, which currently stands as the best upper bound on the stretch factor of the Delaunay graph. However, many researchers believe that the lower bound of $\pi/2$ shown in [9] is also an upper bound on the stretch factor of the Delaunay graph. Even though Delaunay graphs are good planar geometric spanners of Euclidean graphs, however, Delaunay graphs may have unbounded degree. Other geometric (sparse) spanners were also proposed in the literature including the Yao graphs [35], the Θ-graphs [15], and many others (see [22]). However, most of these proposed spanners either do not guarantee planarity, or do not guarantee bounded degree.

Bose et al. [3], [4] were the first to show how to extract a subgraph of the Delaunay graph (and hence it is planar) that is a geometric spanner of the Euclidean graph with stretch factor ≈ 10.02 and degree bounded by 27. In the context of unit disk graphs, Li et al. [19], [20] gave a distributed algorithm that constructs a planar geometric spanner of a unit disk graph with stretch factor C_{del}; however, the spanner constructed can have unbounded degree. Wang and Li [30], [31] then showed how to construct a bounded-degree planar spanner of a unit disk graph with stretch factor $max\{\pi/2, 1 + \pi \sin(\alpha/2)\} \cdot C_{del}$ and degree bounded by $19 + 2\pi/\alpha$, where $0 < \alpha < 2\pi/3$ is a parameter. Very recently, Bose et. al [5] improved the earlier result in [3], [4] and showed how to construct a subgraph of the Delaunay graph that is a geometric spanner of the Euclidean graph with stretch factor: $max\{\pi/2, 1+\pi \sin(\alpha/2)\} \cdot C_{del}$ if $\alpha < \pi/2$ and $(1 + 2\sqrt{3} + 3\pi/2 + \pi \sin(\pi/12)) \cdot C_{del}$ when $\pi/2 \leq \alpha \leq 2\pi/3$, and whose degree is bounded by $14+2\pi/\alpha$. Bose et al. then applied their construction to obtain a planar geometric spanner of a unit disk graph with stretch factor $max\{\pi/2, 1+\pi \sin(\alpha/2)\} \cdot C_{del}$ and degree bounded by $14 + 2\pi/\alpha$, for any $0 < \alpha \leq \pi/3$. This was the best bound on the stretch factor and the degree.

In this paper we develop structural results about Delaunay graphs that allow us to present a very simple linear-time algorithm that, given a Delaunay graph, constructs a subgraph of the Delaunay graph with stretch factor $1 + 2\pi(k \cos(\pi/k))^{-1}$ (with respect to the Delaunay graph) and degree at most k, for any integer parameter $k \geq 14$. This result immediately implies an $O(n \lg n)$ (n is the number of vertices in the graph) algorithm for constructing a planar geometric spanner of a Euclidean graph with stretch factor of $(1 + 2\pi(k \cos(\pi/k))^{-1}) \cdot C_{del}$ and degree at most k, for any integer parameter $k \geq 14$. This significantly improves the previous results of Bose et al. [3], [4], [5], both in terms of the stretch factor and the degree bound. We then show the applications of the results to unit disk graphs by presenting a very simple and strictly-localized distributed algorithm that, given a unit-disk graph embedded in the plane, constructs a planar geometric spanner of the unit disk graph with stretch factor $(1+2\pi(k \cos(\pi/k))^{-1}) \cdot C_{del}$ and degree bounded by k, for any integer parameter $k \geq 14$. These results, in turn, significantly improve all the previous results of [5], [30], [31] on this problem in terms of: the upper bound on the stretch factor, the upper bound on the degree, and the simplicity and locality of the algorithm. In terms of efficiency, the presented algorithm exchanges no more than $O(n)$ messages in total, and runs in $O(\Delta \lg \Delta)$ local time at a node of degree Δ.

To compare our bounds to the previous best bounds in the literature, consider first the problem of computing planar bounded-degree geometric spanners of Euclidean graphs. For a degree bound $k = 14$, our results imply a bound of at most 3.54 on the stretch factor. As the degree bound k approaches ∞, our bound on the stretch factor approaches $C_{del} \approx 2.42$. The very recent results of Bose et al. [5] achieve a lowest degree bound of 17, and that corresponds to a bound on the stretch factor of at least 23. If Bose et al .[5] allow the degree bound to be arbitrarily large (i.e., to approach ∞), their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. In terms of the problem of computing planar bounded-degree geometric spanners of unit disk graphs, the same bounds hold for our construction, i.e., for a degree bound $k = 14$, our results imply a bound of at most 3.54 on the stretch factor, and as k approaches ∞, our bound on the stretch factor approaches C_{del}. The smallest degree bound derived by Bose et al. [5] is 20, and that corresponds to a stretch factor of at least 6.19. If Bose et al. [5] allow the degree bound to be arbitrarily large, then their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. On the other hand, the smallest degree bound derived in Wang et al. [30], [31] is 25, and that corresponds to a bound of 6.19 on the stretch factor. If Wang et al. [30], [31] allow the degree bound to be arbitrarily large, then their bound on the stretch factor approaches $(\pi/2) \cdot C_{del} > 3.75$. Therefore, even the worst bound of at most 3.54 on the stretch factor corresponding to our lowest bound on the degree $k = 14$, beats the best bound on the stretch factor of at least 3.75 corresponding to arbitrarily large degree in both Bose et al. [5] and Wang et al. [30], [31]!

Finally, we show the applications of our results to other families of graphs used for modeling wireless ad-hoc networks. We consider the recently-studied model of quasi unit disk graphs [1], [8], [17] that generalizes unit disk graphs. Given a connected quasi unit disk graph with parameter $0 < d < 1$, we present a strictly-localized distributed algorithm that constructs a geometric spanner of the quasi unit disk graph with maximum degree $O(1/d)$, stretch factor $1 + 2(1 + 2\pi(k \cos\frac{\pi}{k})^{-1}) \cdot C_{del}$, and a bound of $O(1/d)$ on the average number of edges crossing any given edge in the graph.

II. DEFINITIONS AND BACKGROUND

All graphs considered in this paper are graphs embedded in the plane, and all the distances considered are Euclidean distances.

A spanning subgraph H of a graph G is said to have *stretch factor* ρ if for every two points X and Y in G: the ratio of the (Euclidean) length of a shortest path between X and Y in H to the length of a shortest path between X and Y in G is at most ρ. We have the following result from [28].

Lemma 2.1 ([28]): A subgraph H of graph G has stretch factor ρ if and only if for every edge $XY \in G$: the length of a shortest path in H from X to Y is at most $\rho \cdot |XY|$.

For three non-collinear points X, Y, Z in the plane we denote by $\bigcirc XYZ$ the circumscribed circle of triangle $\triangle XYZ$.

A *Delaunay triangulation* of a set of points P in the plane is a triangulation of P in which the circumscribed circle of every triangle contains no point of P in its interior. It is well known that if the points in P are *in general position* (i.e., no four points in P are cocircular) then the Delaunay triangulation of P is unique [10]. In this paper—as in most papers in the literature—we shall assume that the points in P are in general position; otherwise, the input can be slightly perturbed so that this condition is satisfied. The *Delaunay graph* of P is defined as the plane graph whose point-set is P and whose edges are the edges of the Delaunay triangulation of P. It is well known that the Delaunay graph of a set of points P is a spanning subgraph of the Euclidean graph defined on P (i.e., the complete graph on point-set P) whose stretch factor is bounded by $C_{del} = 4\sqrt{3}\pi/9 \approx 2.42$ [15].

Given integer parameter $k > 6$, the *Yao subgraph* [35] of a directed graph G (embedded in the plane) is constructed as follows. At every point M in G, place k equally-separated rays out of M (arbitrarily defined), thus creating k closed cones of size $2\pi/k$ each. Then, the shortest edge in G out of M (if any) in each cone is added to the Yao subgraph of G. We will refer to the placement of k cones around a point M and the selection the shortest edge in each cone by: a *Yao step*. Note that the out-degree of a point in the Yao subgraph of G is bounded by k, but its in-degree may be unbounded.

Two edges MX, MY incident on a point M in a graph G are said to be *consecutive* if one of the angular sectors determined by MX and MY contains no neighbors of $M \in G$.

For simplicity, we will indistinguishably refer to an angular sector formed by two edges MX and MY and its measure by $\angle XMY$. It should be clear from the context whether it is the angular sector or its measure that is being referred to.

III. Bounded Degree Spanners of Delaunay Graphs

Let P be a set of point in the plane and let E be the Euclidean graph defined on point-set P. Let G be the Delaunay graph of P, and note that G is a subgraph of E. This section is devoted to proving the following theorem:

Theorem 3.1: For every integer $k \geq 14$, there exists a subgraph G' of G such that G' has maximum degree k and stretch factor $1 + 2\pi(k \cos \frac{\pi}{k})^{-1}$.

A linear time algorithm that computes G' from G is the key component of our proof. This very simple algorithm essentially performs a *modified Yao step* (see Section II) and selects up to k edges out of every point of G. G' is the spanning subgraph of G consisting of edges chosen by both endpoints.

In order to describe the modified Yao step, we must first develop a better understanding of the structure of the Delaunay graph G. Let CA and CB be edges incident on point C in G such that $\angle BCA \leq 2\pi/k$ and CA is the shortest edge within the angular sector $\angle BCA$. We will show how the above theorem easily follows if, for every such pair of edges CA and CB:

1. we show that there exists a path $\mathcal{P}$ from A to B in G of length $|\mathcal{P}|$, such that: $|CA| + |\mathcal{P}| \leq (1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$, and
2. we modify the standard Yao step to include the edges of this path in G', in addition to including the edges picked by the standard Yao step.

This will ensure that: for any edge $CB \in G$ that is not included in G' by the modified Yao step, there is a path from C to B in G', whose edges are all included in G' by the modified Yao step, and whose cost is at most $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$. We will define below this path and study its structural properties. Then we will modify the standard Yao step accordingly to include edges satisfying these properties.

Lemma 3.2: Let $k \geq 14$ be an integer, and let CA and CB be edges in G such that $\angle BCA \leq 2\pi/k$ and CA is the shortest edge in the angular sector $\angle BCA$. There exists a path $\mathcal{P} : A = M_0, M_1, ..., M_r = B$ in G such that:

(i) $|CA| + \sum_{i=0}^{r-1} |M_i M_{i+1}| \leq (1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|$.

(ii) There is no edge in G between any pair M_i and M_j lying in the closed region delimited by CA, CB and the edges of $\mathcal{P}$, for any i and j satisfying $0 \leq i < j - 1 \leq r$,

(iii) $\angle M_{i-1} M_i M_{i+1} > (\frac{k-2}{k})\pi$, for $i = 1, \cdots, r-1$.

(iv) $\angle CAM_1 \geq \frac{\pi}{2} - \frac{\pi}{k}$.

We break down the proof of the above lemma into two separate cases: (1) when $\triangle ABC$ contains no point of G in its interior, and (2) when there are points of G inside $\triangle ABC$. Define the circle $(O) = \bigcirc ABC$ of center O, and let $\Theta = \angle BCA$. Note that $\angle AOB = 2\Theta \leq 4\pi/k$. Denote by $\overset{\frown}{AB}$ the arc of (O) determined by points A and B and facing $\angle AOB$. The following property about Delaunay graphs can be easily verified by the reader:

Proposition 3.3: If CA and CB are edges of G, then any point of G interior to (O) must be inside the region of (O) delimited by edges BC, CA, and arc $\overset{\frown}{AB}$.

A. The Outward Path

We consider first the case when no points of G are inside $\triangle ABC$. Since both CA and CB are edges in G, by Proposition 3.3, any point of G interior to (O) must be inside the region of (O) delimited by edges BC, CA, and $\overset{\frown}{AB}$. By our assumption in this subsection, no point of G lies inside $\triangle ABC$. It follows that the region of (O) subtended by chord AB that contains C has no points of G in its interior. Keil and Gutwin [15] showed that in this case there exists a path between A and B in G inside the region of (O) subtended by chord AB

that does not contain C, whose length is bounded by the length of $\overset{\frown}{AB}$ (see Lemma 1 in [15]). We sketch their definition in order to illustrate the properties of this path needed for our results. We refer the reader to [15] for the proofs of these properties. For convenience, we give a recursive definition of this path.

1. **Base case:** If $AB \in G$ the path consists of edge AB.
2. **Recursive step:** Otherwise, by the characterization of Delaunay edges [10], at least one point in G is interior to (O), and hence must reside in the region of (O) subtended by chord AB that does not contain C. Let T be such a point with the property that the region of $\bigcirc ATB$ subtended by chord AB that contains T is empty. We call T an *intermediate point* with respect to the pair of points (A, B). Let (O_1) be the circle passing through A and T whose center O_1 lies on segment AO and let (O_2) be the circle passing through B and T whose center O_2 lies on segment BO. Then both (O_1) and (O_2) lie inside (O), and $\angle AO_1T$ and $\angle TO_2B$ are both less than $\angle AOB \leq 4\pi/k$. Moreover, the region of (O_1) subtended by chord AT that contains O_1 is empty, and the region of (O_2) subtended by chord BT and containing O_2 is empty. Therefore, we can recursively construct a path from A to T and a path from T to B, and then concatenate them to obtain a path from A to B.

Definition 3.4: We call the path constructed above the *outward path* between A and B.

Keil and Gutwin [15], from this point on, use a purely geometric argument (with no use of Delaunay graph properties) to show that the length of the obtained path $A = M_0, M_1, \cdots, M_r = B$ (each point M_p, for $p = 1, \cdots, r-1$, is an intermediate point with respect to a pair (M_i, M_j), where $0 \leq i < p < j \leq r$) is smaller than the length of $\overset{\frown}{AB}$. Figure 1 illustrates an outward path between A and B.

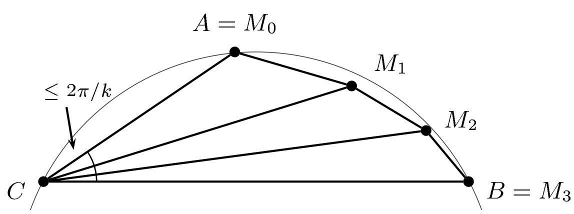

Fig. 1. Illustration of an outward path.

Proposition 3.5: In every recursive step of the outward path construction described above, if M_p is an intermediate point with respect to a pair of points (M_i, M_j), then:

(a) there is a circle passing through C and M_p that contains no point of G, and
(b) circles $\bigcirc CM_iM_p$ and $\bigcirc CM_jM_p$ contain no points of G except, possibly, in the region subtended by chords M_iM_p and M_pM_j, respectively, and not containing C.

Proof: We assume, by induction, that there are circles (O_{M_i}) and (O_{M_j}) passing through C and M_i, and C and M_j, respectively, containing no points of G, and that the circle $(O) = \bigcirc CM_iM_j$ contains no point of G in the interior of the region R' subtended by chord M_iM_j and containing C. (This is certainly true in the base case because $CA, CB \in G$, by proposition 3.3, and by our initial assumptions).

Since M_iM_j is not an edge in G, the point M_p chosen in the construction is the point with the property that the region R of $\bigcirc M_iM_pM_j$ subtended by chord M_iM_j that does not contain C, contains no point of G. Then the circle passing through C and M_p and tangent to $\bigcirc M_iM_pM_j$ at M_p is completely inside $(O_{M_i}) \cup (O_{M_j}) \cup R \cup R'$, and therefore devoid of points of G. This proves part (a).

Finally, the region of $\bigcirc CM_iM_p$ subtended by chord M_iM_p and containing C is inside $(O_{M_i}) \cup R \cup R'$, and therefore contains no point of G in its interior. The same is true for the region of $\bigcirc CM_jM_p$ subtended by chord M_jM_p and containing C, and part (b) holds as well. ■

We are now ready to prove Lemma 3.2 in the case when no point of G lies inside $\triangle ABC$. In this case we define the path in Lemma 3.2 to be the outward path between A and B.

Proof: [Proof of Lemma 3.2 for the case of outward path.]

(i) Let $\Theta = \angle BCA$ and note that $|\overset{\frown}{AB}| = 2\Theta \cdot |OA|$, and $\sin\Theta = \frac{|AB|}{2|OA|}$. Also $|CA| + |\overset{\frown}{AB}|$ is largest when $|CA| = |CB|$, and hence CA and CB are symmetrical with respect to the diameter of $\bigcirc CAB$ passing through C. This latter statement follows from the fact that the perimeter of a convex body is not smaller than the perimeter of a convex body containing it (see page 42 in [2]). In this case we have $\sin\frac{\Theta}{2} = \frac{|AB|}{2|CB|}$. Using elementary trigonometry, it follows from the above facts and from the fact $|CA| \leq |CB|$ that:

$$\begin{aligned} |CA| + |\overset{\frown}{AB}| &\leq |CB| + 2\Theta \cdot |OA| \\ &= |CB| + (\frac{\Theta}{\sin\Theta}) \cdot |AB| \\ &= |CB| + (\frac{\Theta}{\cos\frac{\Theta}{2}}) \cdot |CB| \\ &\leq (1 + 2\pi(k\cos\frac{\pi}{k})^{-1})|CB|. \end{aligned}$$

The last inequality follows from $\Theta \leq 2\pi/k$ and

$k > 2$.

(ii) This part follows directly from part (a) of Proposition 3.5 which implies that CM_p is an edge in G for $p = 0, \cdots, r$, and the fact that G is planar and triangulated.

(iii) If the outward path contains a single intermediate point M_1, then since M_1 lies inside $(O) = \bigcirc CAB$, $\angle AM_1B \geq \pi - \angle AOB/2 \geq \pi - 2\pi/k = (k-2)\pi/k$ (note that $\angle AOB = 2 \cdot \angle ACB$), as desired. Now the statement follows by induction on the number of steps taken to construct the outward path between A and B, using the fact (proved in [15]) that each angle $\angle M_{i-1}O_iM_{i+1}$ at the center of the circle (O_i) defining the intermediate point M_i, is bounded by $\angle AOB$.

(iv) This follows from the fact that $\angle CAM_1 \geq \angle CAB \geq \pi/2 - \pi/k$. The latter inequality is true because $|CA| \leq |CB|$ and $\angle BCA \leq 2\pi/k$ in $\triangle CAB$.

■

B. The Inward Path

We consider now the case when the interior of $\triangle ABC$ contains points of G. Let S be the set of points consisting of points A and B plus all the points interior to $\triangle ABC$ (note that $C \notin S$). Let $CH(S)$ be the points on the convex hull of S. Then $CH(S)$ consists of points $N_0 = A$ and $N_s = B$, and points $N_1, ..., N_{s-1}$ of G interior to $\triangle ABC$. We have the following proposition:

Proposition 3.6: For every $i = 1, \cdots, s-1$:

(a) $|CN_i| \leq |CN_{i+1}|$,

(b) $CN_i \in G$, and

(c) $\angle N_{i-1}N_iN_{i+1} \geq \pi$, where $\angle N_{i-1}N_iN_{i+1}$ is the angle facing point C.

Proof:

(a) This follows from the fact that CA is the shortest edge in its cone, and hence $|CA| \leq |CN_i|$, for $i = 0, \cdots, s$, and the fact that the points $N_0, \cdots, N_s$ are on $CH(S)$ in the listed order.

(b) This follows from the facts: CA and CB are edges in G, $N_0, \cdots, N_s$ are on $CH(S)$, and G is a triangulation.

(c) This follows from the fact that $N_0, \cdots, N_s$ are on $CH(S)$.

■

Since $|CN_i| \leq |CN_{i+1}|$ and no point of G lies inside $\triangle N_iCN_{i+1}$ (N_i and N_{i+1} are on $CH(S)$), CN_i is the shortest edge in the angular sector $\angle N_iCN_{i+1}$. Since $\angle N_iCN_{i+1} \leq \angle BCA \leq 2\pi/k$, by Lemma 3.2 there exists an outward path P_i between N_i and N_{i+1}, for every $i = 0, 1, \cdots, s-1$, satisfying all the properties of Lemma 3.2. Let $A = M_0, M_1, \cdots, M_r = B$ be the concatenation of the paths P_i, for $i = 0, \cdots, r-1$.

Definition 3.7: We call the path $A = M_0, M_1, \cdots, M_r = B$ constructed above the *inward path* between A and B.

Figure 2 illustrates an inward path between A and B.

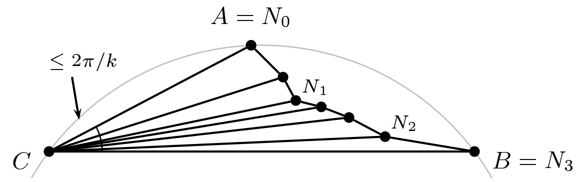

Fig. 2. Illustration of an inward path.

We now prove Lemma 3.2 in the case when there are points of G interior to $\triangle ABC$. In this case we define the path in Lemma 3.2 to be the inward path between A and B.

Proof: [Proof of Lemma 3.2 for the case of inward path.]

(i) Some of the arguments used in the proof of this part are similar in flavor to those used to prove Theorem 7 in [31]. However, the analysis here is tighter and leads to better bounds.

Define A'' to be a point on the half-line $[CA$ such that $|CA''| = |CB|$, and let $(O'') = \bigcirc CA''B$. Denote by α'' the length of the arc of $\bigcirc CA''B$ subtended by chord $A''B$ and facing $\angle A''CB$. For every $i = 0, 1, \cdots, s-1$, we define arc α_i to be the arc of $\bigcirc CN_iN_{i+1}$ subtended by chord N_iN_{i+1} and facing $\angle N_iCN_{i+1}$. For every $i = 0, 1, ..., s-1$, we define N'_i to be the point on the half-line $[CN_i$ such that $|CN'_i| = |CN_{i+1}|$, (O_i) to be the circle $\bigcirc CN'_iN_{i+1}$, and α'_i to be the arc of (O_i) subtended by chord N'_iN_{i+1} and facing $\angle N'_iCN_{i+1}$. Finally, for every $i = 0, \cdots, s-1$, we define N''_i to be the point of intersection of the half-line $[CN_i$ and circle (O''), and α''_i to be the arc of (O'') subtended by chord $N''_iN''_{i+1}$ and facing $\angle N''_iCN''_{i+1}$. By the results in [15], the length of the outward path P_i between N_i and N_{i+1} is bounded by the length of α_i. Since the convex body C_1 delimited by CN_i, CN_{i+1} and α_i is contained inside the convex body C_2 delimited by CN'_i, CN_{i+1} and α'_i, by [2], the perimeter of C_1 is not larger than that of C_2. Denoting by $|P_i|$ the length of path P_i, we get:

$$|P_i| \leq |N_iN'_i| + \alpha'_i, \qquad i = 1, \cdots, s-1. \tag{1}$$

Since (O_i) and (O'') are concentric circles (of center C), and the radius of (O_i) is not larger than that of (O''), we have $\alpha'_i \leq \alpha''_i$, for $i = 0, \cdots, s-1$. It follows from Inequality (1) that:

$$|P_i| \leq |N_iN'_i| + \alpha''_i, \qquad i = 1, \cdots, s-1. \tag{2}$$

Using Inequalities (1) and (2) we get:

$$|CA| + \sum_{i=0}^{s-1} |P_i| \leq |CA| + \sum_{i=0}^{s-1} |N_i N_i'| + \sum_{i=0}^{s-1} \alpha_i''. \quad (3)$$

Noting that $\sum_{i=0}^{s-1} |N_i N_i'| = |CB| - |CA|$ and that $\sum_{i=0}^{r-1} \alpha_i'' = \alpha''$, it follows from Inequality (3) that:

$$\begin{aligned} |CA| + \sum_{i=0}^{s-1} |P_i| &= |CA| + \sum_{i=0}^{r-1} |M_i M_{i+1}| \\ &\leq |CB| + \alpha'' \\ &\leq (1 + 2\pi(k \cos \frac{\pi}{k})^{-1})|CB|. \end{aligned}$$

The last inequality is true by the same argument used in the proof of part (i) in Lemma 3.2 for the case of outward path.

(ii) Since $CN_p \in G$ for $p = 1, \cdots, s-1$ by part (b) of Proposition 3.6, by planarity of G, if such an edge between two points M_i and M_j exists, then M_i and M_j must belong to an outward path between two points N_p and N_{p+1} of $CH(S)$. But this contradicts part (ii) of Lemma 3.2 for the case of the outward path applied to N_p and N_{p+1}.

(iii) For each $i = 0, \cdots, r$, either $M_i = N_j \in CH(S)$, or M_i is an intermediate point on the outward path between two points N_p and N_q in $CH(S)$. In the latter case $\angle M_{i-1} M_i M_{i+1} \geq \angle N_{j-1} M_i N_{j+1} \geq \pi \geq (k-2)\pi/k$ for $k \geq 14$ (N_{j-1} and N_j are points before and after $M_i = N_j$ on $CH(S)$), by part (c) of Proposition 3.6. In the former case $\angle M_{i-1} M_i M_{i+1} \geq (k-2)\pi/k$ by the proof of part (iii) of Lemma 3.2 applied to the outward path between N_p and N_q.

(iv) This follows from $|CA| = |CM_0| \leq |CM_1|$ and $\angle ACM_1 \leq \angle ACB \leq 2\pi/k$, in triangle $\triangle CAM_1$. ∎

C. The Modified Yao Step

In this section we prove Theorem 3.1 by presenting a very simple linear time algorithm that, given a Delaunay graph G, constructs a subgraph G' of G whose degree is bounded by k, and whose stretch factor is bounded by $1 + 2\pi(k \cos \frac{\pi}{k})^{-1}$, for any integer parameter $k \geq 14$. The idea behind the algorithm is very simple: the algorithm ensures that edges in G forming the paths described in Lemma 3.2 are included in G'. Observing that the consecutive edges on such paths form moderately large angles by parts *(iii)* and *(iv)* of Lemma 3.2, the algorithm performs a modified Yao step to ensure that consecutive edges forming large angles are included in G'. The algorithm is described in Figure 3.

Algorithm **Modified Yao step**

INPUT: *A Delaunay graph G; integer $k \geq 14$*
OUTPUT: *A bounded-degree subgraph G' of G*

1. define k disjoint cones of size $2\pi/k$ around every point M in G;
2. in every non-empty cone, select the shortest edge incident on M in this cone;
3. **for** every maximal sequence of $\ell \geq 1$ consecutive empty cones:
   3.1. **if** $\ell > 1$ **then** select the first $\lfloor \ell/2 \rfloor$ unselected incident edges on M clockwise from the sequence of empty cones and the first $\lceil \ell/2 \rceil$ unselected edges incident on M counterclockwise from the sequence of empty cones;
   3.2. **else** (i.e., $\ell = 1$) let MX and MY be the incident edges on M clockwise and counterclockwise, respectively, from the empty cone; **if** either MX or MY is selected **then** select the other edge (in case it has not been selected); **otherwise** select the shorter edge between MX and MY breaking ties arbitrarily;
4. G' is the spanning subgraph of G whose edges are those of G selected by both endpoints.

Fig. 3. The modified Yao Step.

Since the algorithm selects at most k edges incident on any point M, and since the algorithm places in G' only those edges selected by both endpoints, it is clear that each point has degree at most k in G'. It is also clear that since k is a constant, and the total number of edges in the Delaunay graph G is linear in terms of the number of points [10], the algorithm can be implemented to run in linear time. Therefore, what remains to be done in order to prove Theorem 3.1, is to show that for every edge $CB \in G$ such that $CB \notin G'$, there is a path from C to B in G' whose length is at most $\rho = 1 + 2\pi(k \cos \frac{\pi}{k})^{-1}|CB|$. Let $CB \in G$ and suppose that $CB \notin G'$. Then the algorithm did not select CB either out of B or out of C. Without loss of generality, assume that the algorithm did not select CB out of C. Then by step 2 of the algorithm, the algorithm must have selected an edge CA out of C, such that CA and CB are in the same cone, and such that CA is the shortest edge in this cone. Consequently, we have $\angle BCA \leq 2\pi/k$. By Lemma 3.2, there exists a path $\mathcal{P}$: $A = M_0, M_1, \cdots, M_r = B$ between A and B of length $|\mathcal{P}|$ such that $|CA| + |\mathcal{P}| \leq 1 + 2\pi(k \cos \frac{\pi}{k})^{-1}|CB|$. It suffices to show then that all edges on this path are in G'. Equivalently, we will show that all these edges are selected by the algorithm **Modified Yao Step** out of both their endpoints.

Lemma 3.8: The edges CA, $M_i M_{i+1}$ for $i = 1, \cdots, r-1$, are all in G'.

Proof: For brevity, instead of saying that the algorithm **Modified Yao Step** selects an edge MX out of a point M, we will say that M selects edge MX.

By part (iv) of Lemma 3.2, the angle $\angle CAM_1 \geq \pi/2 - \pi/k \geq 6\pi/k$ for $k \geq 14$. Therefore, at least two empty cones must fall within the sector $\angle CAM_1$ determined by the two consecutive edges CA and AM_1, and edges AC and AM_1 will both be selected by A.

Since edge CA is also selected by point C, edge $AC \in G'$.

By part (iii) of Lemma 3.2, for every $i = 1, 2, \cdots, r-1$, the angle $\angle M_{i-1}M_iM_{i+1} \geq (k-2)\pi/k \geq 10\pi/k$ for $k \geq 12$, and hence at least four cones fall within the angular sector $\angle M_{i-1}M_iM_{i+1}$. Since by part *(ii)* of Lemma 3.2 M_iC is the only possible edge inside the angular sector $\angle M_{i-1}M_iM_{i+1}$, it is easy to see that regardless of the position of these four cones with respect to edge M_iC, M_i ends up selecting all edges M_iM_{i-1}, M_iM_{i+1} and M_iC in steps 2 and/or 3 of the algorithm. Since we showed above that A selects edge AM_1, this shows that all edges M_iM_{i+1}, for $i = 0, \cdots, r-2$, are selected by both their endpoints, and hence must be in G'. Moreover, edge $M_{r-1}M_r = M_{r-1}B$ is selected by point M_{r-1}.

We now argue that edge BM_{r-1} will be selected out of B by the algorithm.

First, observe that if $|BM_{r-1}| \leq |\overset{\frown}{AB}| < |CB|$.

Let CD be the other consecutive edge to CB in G (other than CM_{r-1}). Because C does not select B, it follows that $\angle M_{r-1}CD \leq 6\pi/k$. Otherwise, since CM_{r-1} and CB are in the same cone, two empty cones would fall within the sector $\angle BCD$ and C would select B. Since CB is an edge in G, by the characterization of Delaunay edges [10], $\angle CM_{r-1}B + \angle CDB \leq \pi$. By considering the quadrilateral $CDBM_{r-1}$, we have $\angle M_{r-1}CD + \angle DBM_{r-1} \geq \pi$. This, together with the fact that $\angle M_{r-1}CD \leq 6\pi/k$, imply that $\angle DBM_{r-1} \geq (k-6)\pi/k \geq 8\pi/k$, for $k \geq 14$. Therefore, $\angle DBM_{r-1}$ contains at least three cones of size $2\pi/k$ out of B. If one of these cones falls within the angular sector $\angle CBM_{r-1}$ then, since $|M_{r-1}B| < |CB|$, BM_{r-1} must have been selected out of B.

Suppose now that $\angle CBM_{r-1}$ contains no cone inside and hence $\angle CBM_{r-1} < 4\pi/k$. If one of these three cones within sector $\angle DBM_{r-1}$ contains edge CB, then the remaining two cones must fall within $\angle DBC$ and BM_{r-1} will get selected out of B when considering the sequence of at least two empty cones contained within $\angle CBD$. Suppose now that all three empty cones fall within $\angle CBD$. Then we have $\angle CBD \geq 6\pi/k$.

If $\angle M_{r-1}CD \geq 4\pi/k$, then since $M_{r-1}C$ and CB belong to the same cone, the sector $\angle BCD$ must contain an empty cone. Because D is exterior to $\bigcirc CBM_{r-1}$, $\angle CBM_{r-1} < 4\pi/k$, and $\angle M_{r-1}CB \leq 2\pi/k$, it follows that $\angle CDB < \angle M_{r-1}CB + \angle CBM_{r-1} < 6\pi/k < \angle DBC$. Therefore, by considering the triangle $\triangle CDB$, we note that $|CB| < |CD|$. But then edge CB would have been selected by C in step 3 since the sector $\angle BCD$ contains an empty cone, a contradiction.

It follows that $\angle M_{r-1}CD \leq 4\pi/k$, and therefore $\angle M_{r-1}BD \geq (k-4)\pi/k \geq 10\pi/k$ for $k \geq 14$. This means that at least four cones are contained inside sector $\angle DBM_{r-1}$. It is easy to check now that regardless of the placement of the edge BC with respect to these cones, edge BM_{r-1} is always selected out of B by the algorithm. This completes the proof. ■

Since a Delaunay graph of a Euclidean graph of n points can be computed in time $O(n \lg n)$ [10] and has stretch factor C_{del}, we have the following theorem.

Corollary 3.9: There exists an algorithm that, given a Euclidean graph on n points, computes a planar geometric spanner of the Euclidean graph of maximum degree k and stretch factor $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1}) \cdot C_{del}$, where $k \geq 14$ is an integer. Moreover, the algorithm runs in time $O(n \lg n)$.

Corollary 3.9 significantly improves all the previous results in the literature, in particular those in [3], [4], [5].

IV. Geometric Spanners of Unit Disk Graphs

A *unit disk graph* U on a set of n points P in the plane is a graph whose point-set is P, and such that there is an edge between two points X and Y in U if and only if $|XY| \leq 1$. An edge of U is embedded in the plane as the straight-line segment joining its two endpoints. Unit disk graphs are very important because they model wireless networks. We assume that U is connected and that each point in U knows its coordinates through a Global Position System (GPS).

In this section we show how to construct a planar geometric spanner of a unit disk graph having bounded degree and a smaller stretch factor. The results in the previous section do not carry to unit disk graphs because not all the Delaunay graph edges of the point-set P are unit disk edges. However, if we let $Del(U)$ be the Delaunay graph of P, and $UDel(U)$ the subgraph of $Del(U)$ obtained from $Del(U)$ by deleting those edges of length greater than one unit, then it was shown in [19] that $UDel(U)$ is a connected spanning-subgraph of U that is planar and that has stretch factor bounded by C_{del}.

Therefore, if we apply the results in the previous section to $G = UDel(U)$, by observing that all the edges on the path defined in Lemma 3.2 must be unit disk edges (given that edges CA and CB are), it is easy to see that Theorem 3.1 and Corollary 3.9 carry over to unit disk graphs. The only problem, however, is that the construction of $UDel(U)$ cannot be done locally, which is a crucial constraint on any algorithm for wireless networks whose devices could be ad-hoc, and have limited computational resources.

To solve this problem, Wang et al. [19], [20] introduced a subgraph of U which is a supergraph of $UDel(U)$, and that has many desirable properties. We review these definitions next, then we develop some structural results to show how this supergraph defined in [19], [20] can serve as the underlying subgraph G of

U so that the results developed in the previous section carry over to unit disk graphs.

Given a unit disk graph U, the *Gabriel graph* of U, denoted $GG(U)$, is the subgraph of U having the same point-set as U and such that an edge $XY \in U$ is also an edge in $GG(U)$ if and only if the disk of diameter XY contains no points of U other than X and Y [12].

A triangle $\triangle XYZ$ is said to be a *1-localized Delaunay triangle* [19], [20] if $\bigcirc XYZ$ contains no neighbors of X, Y, or Z in its interior. In general, for any integer $i \geq 1$, a triangle $\triangle XYZ$ is said to be an *i-localized Delaunay triangle* [19], [20] if $\bigcirc XYZ$ contains no points of U in its interior that are within i hops from any of the points X, Y, or Z. The *i-localized Delaunay graph* of U, denoted $LDel^{(i)}(U)$, where $i \geq 1$, is the subgraph of U induced by: the set of all Gabriel edges of U plus the edges of all i-localized triangles of U [19], [20].

It was shown in [19], [20] that for any integer $i \geq 2$, $LDel^{(i)}(U)$ is a planar supergraph of $UDel(U)$, and hence has stretch factor bounded by C_{del}. Moreover, the results in [6], [31] show how given U, $LDel^{(2)}(U)$ can be computed by a strictly-localized distributed algorithm exchanging no more than $O(n)$ messages in total (n is the number of points in U), and having a local processing time of $O(\Delta \lg \Delta) = O(n \lg n)$ at a point of degree Δ. Therefore, we will use $LDel^{(2)}(U)$ as the underlying subgraph of U to replace the Delaunay graph G used in the previous section, and we note that: $LDel^{(2)}(U)$ is planar, is a supergraph of $UDel(U)$, and hence has stretch factor C_{del}.

The following lemma appears in [30], [31] and gives a characterization of $LDel^{(2)}(U)$.

Lemma 4.1 ([30], [31]): An edge $XY \in U$ is in $LDel^{(2)}(U)$ if and only if there is a circle passing through points X and Y whose interior contains no point of U within two hops from X or Y.

Let X and Y be two points in the plane and let (O) be any circle passing through X and Y. The chord XY subtends two regions of (O). If Z is a point in the plane different from X and Y, then one of the two regions of (O) subtended by the chord XY is on the same side of XY as Z, whereas the other is on the opposite side of XY as Z. For convenience, we will refer to the former as the region of (O) subtended by XY and *closer* to Z, and to the latter as the region of (O) subtended by XY and *farther* from Z. The following two lemmas can be proved using elementary geometry.

Lemma 4.2: Let X and Y be two points in the plane and let (O) be a circle passing through X and Y. Let Z be any point exterior to (O) and let (O') be $\bigcirc XYZ$. Then the region of (O') subtended by chord XY and farther from Z is inside the region of (O) subtended by XY and farther from Z.

Lemma 4.3: Let $\triangle XYZ$ be a triangle in U and suppose that the region of $\bigcirc XYZ$ subtended by XY and farther from Z contains no points of U that are within two hops from either X or Y. Then the interior of the region of $\bigcirc XYZ$ subtended by XY and farther from Z contains no points of U that are within two hops from Z.

We are now ready to show that Lemma 3.2 holds for $LDel^{(2)}(U)$. For convenience, we will let $G = LDel^{(2)}(U)$ in the remainder of this section.

Let CA and CB be edges in G such that $\angle BCA \leq 2\pi/k$, where $k \geq 14$ is an integer, and CA is the shortest edge in the angular sector $\angle BCA$. We again separate the proof into two parts based on whether $\triangle BCA$ contains points of U or not.

A. The Outward Path

We assume that no points of U are inside $\triangle BCA$. To show that the outward path defined in the previous section carries to G, we have the following structural results.

Lemma 4.4: Let XY and XZ be edges in G and suppose that $ZY \in U$. Then the region of $\bigcirc XYZ$ subtended by XY and farther from Z and the region of $\bigcirc XYZ$ subtended by XZ and farther from Y contain no points of U that are within two hops from X, Y, or Z.

Proof: We will show the statement for the region of $\bigcirc XYZ$ subtended by XY and farther from Z. The statement about the other region follows by symmetry.

The edges in $G = LDel^{(2)}(U)$ are of two types: Gabriel edges and edges in 2-localized Delaunay triangles. We distinguish the following two cases according to the type of edge XY.

Case 1: XY is a Gabriel edge. In this case the circle (O) of diameter XY contains no points of U. In particular, Z is exterior to (O). By Lemma 4.2, the region of $\bigcirc XYZ$ subtended by XY and farther from Z is interior to (O). This shows that the region of $\bigcirc XYZ$ subtended by XY and farther from Z contains no points of U.

Case 2: XY is an edge in a 2-localized Delaunay triangle. By Lemma 4.1, there is a circle (O) passing through X and Y whose interior is empty of any point of U within two hops of X or Y, and by Lemma 4.3 of Z as well. Since Z must be exterior to (O), the region of $\bigcirc XYZ$ subtended by XY and farther from Z is interior to (O), and hence contains no points of U that are within two hops from X, Y, or Z. This completes the proof. ∎

Corollary 4.5: Let XY and XZ be edges in G such that $ZY \in U$. If $ZY \notin G$ then the region of $\bigcirc XYZ$ engulfed by the angular sector $\angle YXZ$ contains a point of U.

Proof: Since $ZY \notin G$, by Lemma 4.1 $\bigcirc XYZ$ must contain a point M that is within 2-hops of either Z

or Y. Since both XY and XZ are in G, by Lemma 4.4, the region of $\bigcirc XYZ$ subtended by XY and farther from Z and the region of $\bigcirc XYZ$ subtended by XZ and farther from Y contain no points of U that are within two hops from Y or Z. It follows that the region of $\bigcirc XYZ$ engulfed by the angular sector $\angle YXZ$ must contain a point of U that is within 2-hops of Y or Z. This completes the proof. ■

Lemma 4.6: Let CA and CB be edges in G such that $AB \in U$, and let $(O) = \bigcirc ABC$. Suppose that CA and CB are on one side of the diameter passing through C in (O). Suppose further that no point of G is interior to $\triangle CAB$. Let R be the region of of (O) subtended by AB and containing C. Then for any point M in the region of (O) subtended by AB that does not contain C, R is devoid of any point within two hops from M.

Proof: Since both CA and CB are in G, $AB \in U$, and no point of G is interior to $\triangle CAB$, it follows from Lemma 4.4 that the region R is devoid of any point within two hops of A, B, or C. Since CA and CB are on one side of the diameter passing through C in (O), any point M in the region subtended by AB that does not contain C must be a neighbor of C, A, and B in U. Now using Lemma 4.3 and elementary geometry, it is easy to verify that R does not contain any point within two hops from M. ■

Lemma 4.7: Let CA and CB be edges in G such that $AB \in U$, and let $(O) = \bigcirc ABC$. Suppose that CA and CB are on on opposite sides of the diameter passing through C in (O). Suppose further that no point of G is interior to $\triangle CAB$, and let R be the region of of (O) subtended by AB and containing C. Then no point of G is interior to R.

Proof: Since both CA and CB are in G and no point of G is interior to $\triangle CAB$, it follows from Lemma 4.4 that the region R is devoid of any point within two hops of A, B, or C. Since CA and CB are on opposite sides of the diameter passing through C in (O), any point in R must be adjacent to C, in U (since either CA or CB is a longest chord in R). It follows that no point of G is interior to R. ■

Lemma 4.8: Let CA, CB be edges in G such that $\angle BCA \leq 2\pi/k$, where $k \geq 14$ is an integer. Then the outward path described by the recursive construction in the previous section is well defined.

Proof: It suffices to show that in each recursive step, if there is no edge between points M_iM_j then the intermediate point M_p for pair (M_i, M_j) is well defined. Let $(O) = \bigcirc BCA$, and let R be the region of (O) subtended by AB and containing C. We distinguish two cases. (Note that since CA and CB are in $G \subseteq U$, and $\angle BCA \leq 2\pi/k \leq \pi/7$, it follows that $AB \in U$.)

If CA and CB are on one side of the diameter of (O) through C, then by Lemma 4.6, the interior of R is empty of any point of G within two hops from any point M in the region of (O) subtended by AB and not containing C. It follows from Corollary 4.5 that, for any pair of points (M_i, M_j) in the recursive construction (initially $(M_i, M_j) = (A, B)$), either $M_iM_j \in G$ or the region of the circle defined in the recursive construction for pair (M_i, M_j) (initially this circle is (O) for pair (A, B)) contains an intermediate point M_p in its region subtended by M_iM_j and not containing its center. Therefore, each intermediate point described by the recursive construction section is well defined.

If CA and CB are on opposite sides of the diameter of (O) through C, then by Lemma 4.7, the interior of R is empty of any points in G. By a similar argument to the one made in the above paragraph, each intermediate point described by the recursive construction section is well defined.

■

By Lemma 4.8, for any edge CA and CB in G such that $\angle BCA \leq 2\pi/k$ ($k \geq 14$), the outward path between A and B as defined in the previous section is also in G, and its length satisfies the same bound as before. This establishes part (i) in Lemma 3.2. Moreover, since the proofs of parts (iii) and (iv) in the lemma rely solely on geometric arguments, parts (iii) and (iv) hold as well. To show part (ii), we first need the following lemmas.

Lemma 4.9: Let CA and CB be edges in G such that $AB \in U$, and let $(O) = \bigcirc ABC$. Suppose that CA and CB are on one side of the diameter passing through C in (O). Suppose further that no point of G is interior to $\triangle CAB$. In every recursive step of the outward path construction described above, if M_p is an intermediate point with respect to the pair of points (M_i, M_j), then there is a circle passing through C and M_p that contains no point of G within two hops from C or M_p.

Proof: Noting that the points in the region of (O) delimited by CA, CB, and $\widehat{AB}$ form a clique in U, the proof is exactly the same as that of Lemma 3.5 using Lemma 4.4. ■

Lemma 4.10: Let CA and CB be edges in G such that $\angle BCA \leq 2\pi/k$, where $k \geq 14$ is an integer, and let $(O) = \bigcirc ABC$. Suppose that CA and CB are on opposite sides of the diameter passing through C in (O). Suppose further that no point of G is interior to $\triangle CAB$. If M_p is an intermediate point with respect to the pair of points (M_i, M_j), then there is a circle passing through C and M_p that contains no point of G within two hops from C or M_p.

Proof: Note that in this case there can be points M_p such that CM_p is not an edge in U. The proof is similar to that of Lemma 3.5, and uses the fact that $\angle BCA \leq 2\pi/k$. The proof is a bit technical but rely purely on elementary geometric and trigonometric arguments. ■

Now we are ready to prove part (ii) of Lemma 3.2. Suppose that there is an edge M_iM_j with $0 \leq i < j-1 \leq r$. Then there must be an intermediate point M_p with respect to a pair (M_q, M_t) (with $0 \leq q \leq i < p < j \leq t \leq r$) that was found in the recursive construction earlier. This means that edge M_iM_j will intersect (or share endpoints) with segments M_qM_p and M_tM_p. From the statements of Lemma 4.9 and Lemma 4.10 applied to the pair (M_q, M_t) and the intermediate point M_p, and noting that C and M_p are both within two hops from M_i and M_j (note that $CA \in U$ and all the points in the region of $\bigcirc CBA$ subtended by BA that does not contain C are neighbors of A because $\angle BCA \leq 2\pi/k \leq \pi/7$ when $k \geq 14$), we conclude that any circle passing through M_i and M_j will either contain C or M_p, contradicting the assumption that M_iM_j is an edge of G.

B. The Inward Path

We assume that there are points of U interior to $\triangle CBA$.

Now that the outward path definition from the previous section carries over, the inward path is defined in the same way as in the previous section. Again, parts (i), (iii), and (iv) of Lemma 3.2 follow immediately. To show part (ii), we first present the following lemma.

Lemma 4.11: Let $(A = N_0, \cdots, N_s = B)$ be the points on $CH(S)$ as defined in Subsection III-B. Then CN_i is an edge in $G = LDel^{(2)}(U)$ for $i = 0, \cdots, s$.

Proof: We know that $CN_0 = CA$ and $CN_s = CB$ are edges in G.

By property (a) in Proposition 3.6, we have $|CA| = |CN_0| \leq |CN_1| \leq \cdots \leq |CN_s| = |CB|$. Since $CN_0 = CA \in G$, it suffices to show that $CN_1 \in G$ and the argument can be repeated with CN_i for $i = 2, \cdots, s-1$ (using $CN_{i-1} \in G$).

To show that $CN_1 \in G = LDel^{(2)}(U)$, by Lemma 4.1, it suffices to show that there exists a circle (O) passing through C and N_1 whose interior contains no point of U within two hops of C or N_1. Let (O) be the circle passing through C and N_1 and tangent to the straight line N_0N_1. If (O) intersects CN_0, let C' be the point of intersection other than C, and observe that since (O) is tangent to N_0N_1, C' must be interior to the segment CN_0. If (O) intersects CB, let C'' be the point of intersection other than C, and observe that C'' must be interior to segment CB. The latter statement follows from the fact that the points $(A = N_0, \cdots, N_s = B)$ lie on $CH(S)$ and $|CA| = |CN_0| \leq |CN_1| \leq \cdots \leq |CN_s| = |CB|$. The circle (O) can be partitioned into at most four parts depending on whether this circle intersects CN_0 and CB or not: (1) the part subtended by CC' and farther from N_1, (2) the part engulfed by the angular sector $\angle C'CN_1$, (3) the part engulfed by the angular sector $\angle N_1CB$, and (4) the part subtended by CC' and farther from N_1.

Since $(N_0, \cdots, N_s)$ lie on $CH(S)$, and from the fact that $|CA| = |CN_0| \leq |CN_1| \leq \cdots \leq |CN_s| = |CB|$, the regions defined in (2) and (3) above lie inside the region enclosed by $\triangle CAB$ and the path $(N_0, \cdots, N_s)$, and hence contain no point of U. In particular, these two regions contain no point within 2-hops of C or N_1.

To show that the region in (1) contains no point with 2-hops of C or N_1, we show two claims. First, we show that this region is contained inside the region of $\bigcirc CN_1N_0$ subtended by CN_0 and farther from N_1, and second that the latter region contains no point within 2-hops of C or N_1. We proceed to show these two claims next.

To show the first claim, note that since N_0 is exterior to (O), by Lemma 4.2 the region of $\bigcirc CN_1N_0$ subtended by CN_1 and farther from N_0 is contained within the region of (O) subtended by CN_1 and farther from N_0. Equivalently, the region of (O) subtended by CN_1 and closer to N_0 is contained within the region of $\bigcirc CN_1N_0$ subtended by CN_1 and closer to N_0. It follows from this that the region of (O) subtended by CC' and farther from N_1, which is the region in part (1) above, is contained within the region of $\bigcirc CN_1N_0$ subtended by CN_0 and farther from N_1.

To show the second claim, note that since $CN_0 \in G$, by Lemma 4.1 there exists a circle (O') passing through C and N_0 that contains no point of U within 2-hops of C or N_0. In particular, N_1 is exterior to the circle (O'), and by Lemma 4.2 the region of $\bigcirc CN_1N_0$ subtended by CN_0 and farther from N_1 is contained with the region of (O') subtended by CN_0 and farther from N_1. Therefore, the region of $\bigcirc CN_1N_0$ subtended by CN_0 and farther of N_1 contains no point within 2-hops of C or N_0, and by Lemma 4.3, no neighbor within two hops from N_1 (note that CN_0, CN_1 and N_0N_1 are all edges in U). Therefore, the region of $\bigcirc CN_1N_0$ subtended by CN_0 and farther of N_1 contains no point within two hops from C or N_1.

It follows that the region defined in part (1) above contains no point within two hops of C or N_1.

Similar to the above, and using the fact that $CB \in G$, we can show that the region in (3) contains no point within two hops from C or N_1.

It follows that the circle (O) contains no point within two hops of either C or N_1, and $CN_1 \in G$. ∎

Now part (ii) of Lemma 3.2 follows from Lemma 4.11, part (ii) of Lemma 3.2 for the outward path, and the planarity of G.

C. The Algorithm

The algorithm is basically the same as the **Modified Yao Step** algorithm in Section III-C, only presented in a distributed strictly-localized fashion. Each point $M \in U$ performs the algorithm given in Figure 4.

Algorithm **Spanner Constructor**

INPUT: $G = LDel^{(2)}(U)$; *integer* $k \geq 14$
OUTPUT: G': *a bounded degree planar geometric spanner of* U

1. M places k identical cones each of size $2\pi/k$ around itself;
2. **for** every non-empty cone M selects the shortest edge in the cone;
3. **for** every maximal sequence of $\ell \geq 1$ consecutive empty cones M does the following:
   3.1. **if** $\ell > 1$ **then** M selects the first $\lfloor \ell/2 \rfloor$ unselected incident edges clockwise from the sequence of empty cones and the first $\lceil \ell/2 \rceil$ unselected incident edges counterclockwise from the sequence of empty cones;
   3.2. **else** (i.e., $\ell = 1$) let MX and MY be the incident edges clockwise and counterclockwise, respectively, from the empty cone; **if** either MX or MY is selected **then** M selects the other edge (in case it has not been selected); otherwise M selects the shorter edge between MX and MY breaking ties arbitrarily;
4. M sends a message to every neighbor X notifying it of whether M selected the edge MX or not.

Upon receiving a message from a neighbor N, M performs the following steps:

1. decide the status of the edge MN as follows: $MN \in G'$ if and only if MN has been selected by both M and N.
2. **if** for every neighbor X the status of the edge MX has been determined **then** M finishes processing.

Fig. 4. The algorithm Spanner Constructor.

The spanner G' of U consists of those edges in G selected by both their endpoints.

The results in [6], [31] show how $G = UDel^{(2)}(U)$ can be constructed by a distributed strictly-localized algorithm that exchanges no more than $O(n)$ messages in total (each of length $O(\lg n)$ bits), where n is the number of points in U, and a local processing time of $O(\Delta \lg \Delta) = O(n \lg n)$ at a point of degree Δ. Noting that in the algorithm **Spanner Constructor** each point needs only to notify its neighbors about its set of selected edges, the communication cost of the algorithm **Spanner Constructor** is $O(n)$ messages. Moreover, since the number of cones around any point M is bounded a constant k, it is easy to see that all steps in **Spanner Constructor** can be performed by M of degree Δ in $O(\Delta \lg \Delta) = O(n \lg n)$ local processing time by first sorting its set of incident edges in G in counterclockwise (or clockwise) order. It follows that the local processing time of any point of degree Δ during the whole construction of the spanner, including the construction of G, is $O(\Delta \lg \Delta) = O(n \lg n)$.

The statement of Lemma 3.8 in the previous section holds true as well (the proof is exactly the same) for the algorithm **Spanner_Constructor**. Summarizing all the results in this section, we conclude with the following theorem.

Theorem 4.12: There exists a distributed strictly-localized algorithm that, given a unit graph on n points, computes a planar geometric spanner of the unit disk graph of maximum degree k and stretch factor $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1}) \cdot C_{del}$, for any integer $k \geq 14$. Moreover, the algorithm exchanges no more than $O(n)$ messages in total, and have a local processing time of $\Delta \lg \Delta$ at a point of degree Δ.

Theorem 4.12 significantly improves all the previous results in the literature [5], [30], [31].

V. GEOMETRIC SPANNERS OF QUASI-UNIT DISK GRAPHS

Even though unit disk graphs are commonly used to model wireless networks, in reality, they might deviate from many real wireless networks due to reasons including multi-path fading [13], [27], antenna design issues, inaccurate node position estimation, etc. A more general network model, the *quasi unit disk graph*, has been recently proposed to capture the nonuniform characteristics of wireless networks. Formally, this model is defined as follows.

Let P be set of points in the plane and let $0 \leq d \leq 1$ be a constant. A *quasi unit disk graph* on P with parameter d, denoted $Quasi$-P, is defined as follows: for any two point X and Y in P, XY is an edge in $Quasi$-P if $|XY| \leq d$, and XY is not an edge in $Quasi$-P if $|XY| > 1$. If $d < |XY| \leq 1$ then XY may or may not be an edge in $Quasi$-P.

The quasi unit disk graph model was first studied in [1] and further developed in [17]. However, the work in [1], [17] focused more on routing algorithms for the case when $d \geq \sqrt{2}/2$. Separability and other topology-control issues of quasi unit disk graphs were also studied in [8].

In this sections we show how the results developed in the previous sections extend to the quasi unit disk graph model. Let $Quasi$-P be a quasi unit disk graph with parameter $0 < d \leq 1$ on a set of n points P, and assume that $Quasi$-P is connected. We use the same approach used in [8] to construct power spanners of quasi unit disk graphs. Note that a power spanner of a graph is not necessarily a geometric spanner, and hence the algorithm in [8] for constructing power spanners of unit disk graphs cannot be used for constructing geometric spanners.

Call an edge XY in $Quasi$-P a *short* edge if and only if $|XY| \leq d$; otherwise, call XY a *long* edge. Let E_{short} the set of short edges, and note that the graph induced by E_{short} is a unit disk graph with unit d. Denote this graph by U_{short} and note that U_{short} may not be connected.

Apply the algorithm **Spanner_Constructor** to each component of U_{short} to construct a planar geometric spanner of this component of degree bounded by k and stretch factor $(1 + 2\pi(k \cos \frac{\pi}{k})^{-1}) \cdot C_{del}$, where $k \geq 14$ is an integer parameter. Add all the edges in the spanners of the components to G'.

Now impose a grid of cell-size $\frac{d}{\sqrt{2}} \times \frac{d}{\sqrt{2}}$ on the plane. Note that any two points in the same cell are connected in $Quasi$-P and that any long edge must connect points in two different cells. For each pair of cells, add to G' the shortest edge between those two cells (i.e., the shortest edge having one endpoint in one of the two cells and the other endpoint in the other cell). Observe that determining the shortest edge between two cells can be done in a strictly-localized fashion since the points in a cell form a clique. This completes the construction of G'.

We have the following theorem whose proof is very similar to the proof of Theorem 5 in [8] for the case of power spanners.

Theorem 5.1: Let $Quasi$-P be a connected quasi unit disk graph on n points with parameter $0 < d \leq 1$. For any integer $k \geq 14$, there is a strictly-localized distributed algorithm that constructs a geometric spanner of $Quasi$-P with the following properties: (1) its maximum degree is $O(1/d)$, (2) its stretch factor is $1 + 2(1 + 2\pi(k \cos \frac{\pi}{k})^{-1}) \cdot C_{del}$, and (3) the average number of edges crossing any given edge in $Quasi$-P is $O(1/d)$.

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