# Crossing Number of Graphs with Rotation Systems


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**Abstract**

We show that computing the crossing number of a graph with a given rotation system is **NP**-complete. This result leads to a new and much simpler proof of Hliněný's result, that computing the crossing number of a cubic graph (no rotation system) is **NP**-complete.


# 1 Introduction

Computing the crossing number is **NP**-complete, as shown by Garey and Johnson [2]. Hliněný recently showed, using a rather complicated construction, that even determining the crossing number of a cubic graph is **NP**-complete [3], a long-standing open problem.

We investigate a new approach to cubic graphs through graphs with rotation systems. We show that determining the crossing number of a graph with a given rotation system is **NP**-complete, and then prove that this problem is equivalent to determining the crossing number of a cubic graph. This also gives a new and easy proof that determining the minor-monotone crossing number (defined in [1]) is **NP**-complete.

# 2 NP-hardness

Consider a graph drawn in the plane (or any orientable surface). The *rotation* of a vertex is the clockwise order of its incident edges. A *rotation system* is the list of rotations of every vertex. We are interested in drawings of a graph in the plane with a fixed rotation system, which has appeared in previous papers [4, 5].

We also consider allowing "flipped" rotations (previously seen in [4]). Given a rotation of a vertex $v$, the *flipped rotation* reverses the cyclic order of the edges incident to $v$.

**Theorem 2.1** *Computing the crossing number of a graph with a given rotation system is* **NP***-complete. The problem remains* **NP***-complete if we allow the rotation at each vertex to flip independently.*

**Proof** We adapt Garey and Johnson's reduction from OPTIMAL LINEAR ARRANGEMENT to CROSSING NUMBER [2]. Given a graph $G = (V, E)$, a *linear arrangement* is an injective function $f : V \to 1, \ldots, |V|$, and the *value* of the arrangement is computed as

$$\sum_{uv \in E} |f(u) - f(v)|.$$

Given $G$ and $k$, deciding whether $G$ allows a linear arrangement of value at most $k$ is **NP**-complete [2, GT42].

Let us fix a connected graph $G = (V, E)$, with $V = v_1, \ldots, v_n$, $m = |E|$, and $k$. We may assume that $n \leq m$. From $G$ we construct an edge-weighted graph $H$ with fixed rotation system, as shown in Figure 1. The use of weighted edges simplifies the construction; later we will replace each weighted edge by a small unweighted graph, obtaining a simple graph $H'$ with a fixed rotation system. Note that for a fixed drawing of a weighted graph, a crossing of an edge of weight $k$ with an edge of weight $l$ contributes $kl$ to the crossing number.

We start with a cycle $(u_1, \ldots, u_{4n})$, and a single vertex $u_0$ connected to each vertex on the cycle. We choose the edge-weights of this part of the graph so high that it has to be embedded without any intersections.

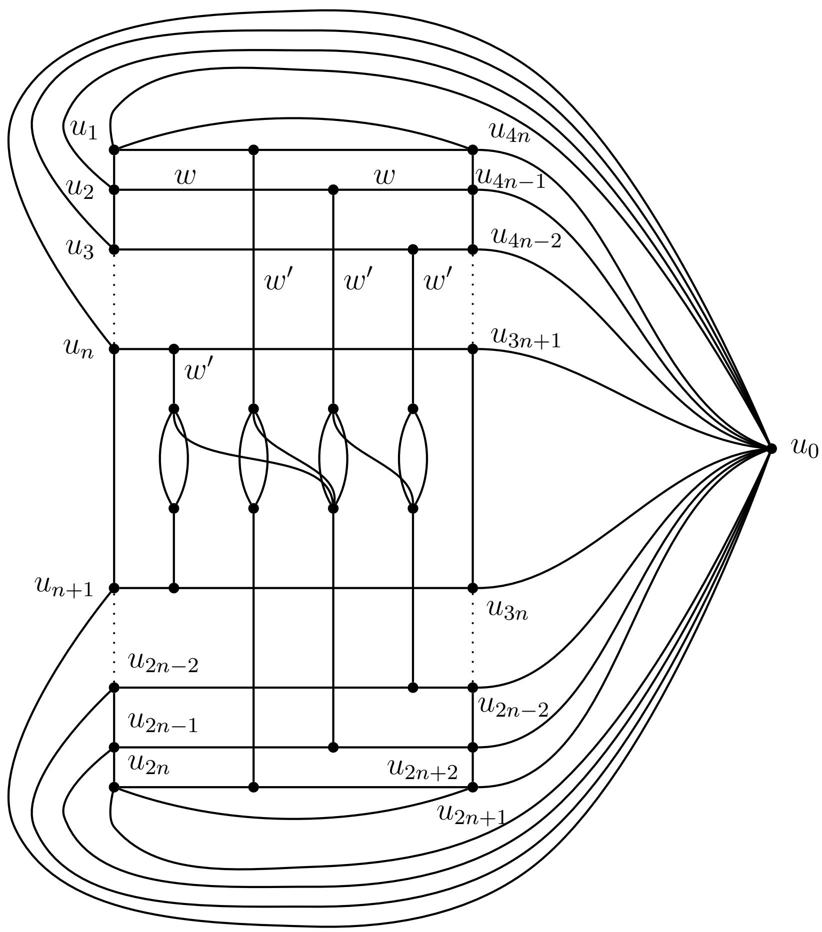


Figure 1: The graph $H$

For every $1 \leq i \leq 2n$ we connect $u_i$ to $u_{4n+1-i}$ by a path $P_i$ of length 2 and edges of weight $w$. Furthermore, we connect the midpoints of $P_i$ and $P_{2n+1-i}$ by a path $Q_i$ of length 3 with edges of weight $w'$, whose middle edge $a_i b_i$ has been replaced by two edges of weight $w'/2$ ($1 \leq i \leq n$).

Finally, we encode $G$ as follows: for each edge $v_i v_j \in E$ we add an edge from $a_i$ to $b_j$ (with $i < j$, an arbitrary choice). The rotation of $H$ is as shown in Figure 1. At $a_i$, each edge from $E$ is inserted into the rotation at $a_i$ between the two $a_i, b_i$-edges of weight $w'/2$; we do likewise at every $b_i$. The edges of $E$ at $a_i$ can be ordered arbitrarily (same at $b_i$).

This concludes the description of $H$. We let $k' = n(n-1)ww' + kw' + m^2$, where $w = 5m^4$ and $w' = 2m^2$. We claim that $G$ allows a linear arrangement of value at most $k$ if and only if $H$ (with the rotation system shown in the drawing) has crossing number at most $k'$.

If $G$ has a linear arrangement of value at most $k$, we can draw $H$ using the order of the $v_i$ in that linear arrangement to obtain a drawing of crossing number at most $k'$ (the $m^2$ term compensates for the potential pairwise crossings of the edges in $H$ that represent edges in $E$).

For the reverse implication, consider a drawing of $H$ with crossing number at most $k' = n(n-1)ww' + kw' + m^2$. Then $k' < n^2ww' + m^2w' + m^2$, and by choice of $w$ and $w'$ this is at most $10m^8 + 2m^4 + m^2 < w^2$. Hence, in our drawing, no two edges of weight $w$ intersect each other, and, therefore, the paths $P_i$ ($1 \leq i \leq 2n$) are drawn as shown in Figure 1.

Next, consider the modified paths $Q_i$. $Q_i$ must intersect each of the paths $P_{i+1}$ through $P_{2n-i}$, contributing $(2n-2i)ww'$ to the crossing number. Summing these values for $i = 1, \ldots, n$, we observe a contribution of at least $n(n-1)ww'$ by intersections between the $Q_i$ and the $P_i$ to the crossing number. This leaves $k' - n(n-1)ww' = kw' + m^2 < m^2w' + m^2 < w'w' < w'w$ crossings, implying that there cannot be any further intersections between a $Q_i$ and a $P_i$ (since it would contribute $w'w$ to the crossing number, more than is left). By the same reasoning, we also do not have intersections between any two $Q_i$.

Finally, we want to argue that all the $a_i$ and $b_i$ lie between $P_n$ and $P_{n+1}$. Since $Q_n$ lies entirely between $P_n$ and $P_{n+1}$ (as we argued earlier), so do $a_n$ and $b_n$. Consider any $a_i$ or $b_i$. As $G$ is connected by assumption, there is a path from $a_n$ to $a_i$ using edges encoding $G$ and edges of weight $w'/2$. If this path intersects $P_n$ or $P_{n+1}$, it contributes $w$ or more to the crossing number. However, since $k' - n(n-1)ww' = kw' + m^2 < m^2w' + m^2 = 2m^4 + m^2 < 5m^4 = w$, this is not possible. Therefore, $a_i$ and $b_i$ are also located between $P_n$ and $P_{n+1}$.

In summary, the drawing of $H$ looks as shown in Figure 1. This drawing clearly indicates a linear arrangement $f$ of $G$. An edge $e = uv$ contributes at least $|f(u) - f(v)|w'$ to the crossing number of $H$, so $\sum_{uv \in E} |f(u) - f(v)| \leq kw' + m^2$. Since $m^2 < w$, the value of the linear arrangement is at most $k$.

The last step is to replace each edge $e$ of weight $x$ by $x$ parallel edges, and then subdivide each of those edges: the effect is that $e$ is replaced by a copy of $K_{2,x}$ with the endpoints of $e$ identified with the partite set of size 2. The new edges are inserted in the rotation at where $e$ was, and the new edges are ordered as indicated in Figure 2. Thus we obtain an unweighted graph $H'$ from $H$. Since we can draw any of the parallel edges alongside whichever one is involved in the smallest number of crossings, we may assume that an optimal drawing of $H'$ has all parallel edges routed in parallel; also, subdivisions do not affect the crossing number. Therefore, $\mathrm{cr}(H') = \mathrm{cr}(H)$, and $H'$ is an unweighted graph with fixed rotation system for which is it is **NP**-hard to determine crossing number.

Note that the argument showing that the drawing of $H$ looks as in Figure 1 did not make any assumptions about the rotation at a vertex. Therefore, even if we allow flipped rotations, we can still conclude that the drawing of $H$ yields a linear arrangement of value at most $k$. Consequently, computing the crossing number of graphs with rotation systems remains **NP**-complete if we allow rotations to flip.

$\square$

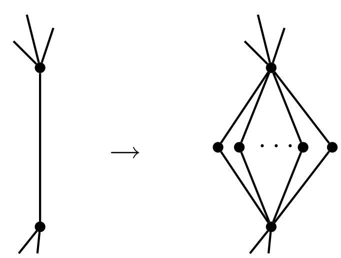

Figure 2: Replacing an edge by parallel paths

# 3 Cubic Graphs

In this section we show how to use Theorem 2.1 to prove that computing the crossing number of a cubic graph is **NP**-complete. This was a long-standing open question that was solved only recently by Petr Hliněný, using a rather complicated construction.

**Theorem 3.1 (Hliněný [3])** *Computing the crossing number of a cubic graph is* **NP***-complete.*

**Proof** Consider a graph $G$ with a given rotation system, and let $\mathrm{cr}(G)$ be the minimum number of crossings in a drawing of $G$ such that every vertex has either the given rotation, or that rotation flipped. We will construct a cubic graph $G'$ such that $\mathrm{cr}(G) \leq k$ if and only if $\mathrm{cr}(G') \leq k$.

Replace each vertex $v$ by a grid of hexagons $H_v$, made up of $2k+1$ rows of $d = \deg(v)$ hexagons per row. Let the vertices along the top be labeled $v_1, \ldots, v_{2d+1}$, as shown in Figure 3.

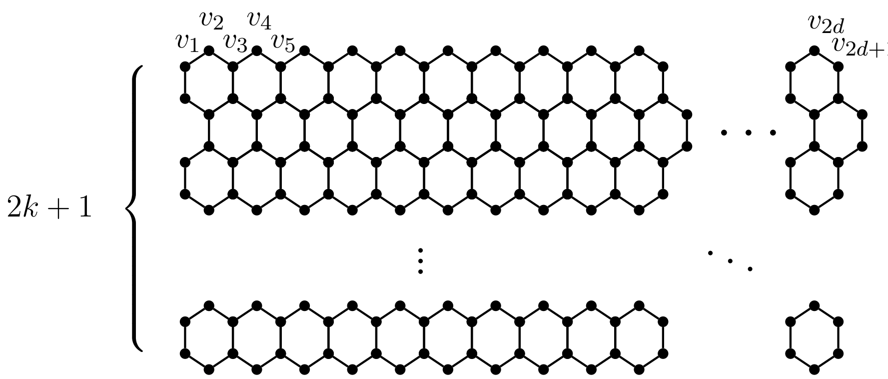


Figure 3: Hexagonal grid replacing vertex

Let us say the rotation at $v$ lists edges in order $e_1, \ldots, e_d$ (cyclic order, so the first element is chosen arbitrarily). We can then make each $e_i$ incident to $v_{2i}$. Repeating this at every vertex, we obtain a graph $G'$ of maximum degree 3.

Suppose that we have a drawing of $G'$ with at most $k$ crossings. Let $X$ be the set of edges of $H_v$ involved in crossings. Of the $2k+1$ rows of hexagons, $k+1$ have odd index, and any two rows with distinct odd indices are disjoint. Since $|X| \leq k$, there is a row of hexagons $R_v$ in $H_v - X$. The drawing of $R_v$ has $k+1$ faces, and clearly all but one is empty. Without loss of generality, each inner face is nonempty (and bounded by a hexagon).

For each $1 \leq i \leq d$, there is a unique "vertical" path $P_i$ in $H_v$ from the endpoint of $e_i$ in $H_v$ to $R_v$. Let $H'_v$ be the union of $R_v$ and all $P_i$. Consider the restriction of the drawing of $G'$ to the drawings of the

$H'_v$ (for all $v \in V$) and all edges between distinct $H_v$ (the edges in $E$). At each $H_v$, consider the edges $e_i$ extended through the paths $P_i$ until they reach $R_v$. Since $R_v$ is not involved in any intersections, the ends of these extended paths at $R_v$ are in the order of the rotation of $v$, or the reverse of that order. Hence, if we contract every $R_v$ to a single point, we obtain a subdivision of $G$ with the given rotation or flipped rotation at each vertex of $G$. Removing the subdivisions obtains the desired drawing of $G$. Since none of the operations (restriction, contraction of crossing-free edges, removing subdivisions) increase the crossing number we have obtained a drawing of $G$ with crossing number at most $k$.

Thus, computing the crossing number of a graph of maximum degree 3 is **NP**-complete.

Finally, given a graph of maximum degree 3, we can easily modify it to get a cubic graph with the same crossing number: repeatedly delete all vertices of degree less than 2, and for each vertex $v$ of degree 2, add the graph shown in Figure 4 to $G'$. □

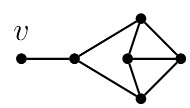


Figure 4: Standard gadget for cubic graphs

As Hliněný observes, this result also implies that computing the minor-monotone crossing number is **NP**-complete [3].

Our Theorem 2.1 is in turn derivable from Hliněný's result, as the gadget in Figure 5 shows.

If we take a cubic graph and replace each vertex by the gadget, we obtain a graph with a fixed rotation system, whose crossing number differs from the crossing number of the original graph by an additive term.

# References


[1] Gašper Fijavž Drago Bokal and Bojan Mohar. Minor-monotone crossing number. In Stefan Felsner, editor, *2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05)*, volume AE, pages 123–128. Discrete Mathematics and Theoretical Computer Science, 2005.

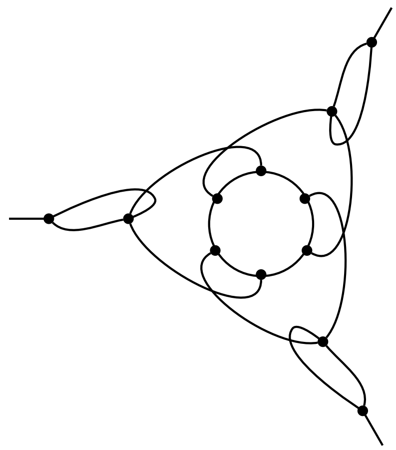

Figure 5: Rotation gadget


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