Odd Crossing Number Is Not Crossing Number

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Abstract

The *crossing number* of a graph is the minimum number of edge intersections in a plane drawing of a graph, where each intersection is counted separately. If instead we count the number of pairs of edges that intersect an odd number of times, we obtain the *odd crossing number*. We show that there is a graph for which these two concepts differ, answering a well-known open question on crossing numbers. To derive the result we study drawings of maps on the annulus.

1 A Confusion of Crossing Numbers

Intuitively, the crossing number of a graph is the smallest number of edge crossings in any plane drawing of the graph. As it turns out, this definition leaves room for interpretation, depending on how we answer the questions: what is a drawing, what is a crossing, and how do we count crossings? The papers by Pach and Tóth [4] and Székely [5] discuss the historical development of various interpretations and, often implicit, definitions of the crossing number concept.

A *drawing* D of a graph G is a mapping of the vertices and edges of G to the Euclidean plane, associating a distinct point with each vertex, and a simple plane curve with each edge such that the ends of an edge map to the endpoints of the corresponding curve. For simplicity, we also require that

- a curve does not contain any endpoints of other curves in its interior,
- two curves do not touch (that is, intersect without crossing), and
- no more than two curves intersect in a point (other than at a shared endpoint).

In such a drawing the intersection of the interiors of two curves is called a *crossing*. Note that by the restrictions we placed on a drawing, crossings do not involve endpoints, and at most two curves can intersect in a crossing. We often identify a drawing with the graph it represents.

Given a drawing D of a graph G in the plane we define

- $\mathrm{cr}(D)$ - the total number of crossings in D;
- $\mathrm{pcr}(D)$ - the number of pairs of edges which cross at least once; and
- $\mathrm{ocr}(D)$ - the number of pairs of edges which cross an odd number of times.

Remark 1 For any drawing D, we have $\mathrm{ocr}(D) \leq \mathrm{pcr}(D) \leq \mathrm{cr}(D)$.

We let $\mathrm{cr}(G) = \min \mathrm{cr}(D)$, where the minimum is taken over all drawings D of G in the plane. We define $\mathrm{ocr}(G)$ and $\mathrm{pcr}(G)$ analogously.

Remark 2 For any graph G, we have $\mathrm{ocr}(G) \leq \mathrm{pcr}(G) \leq \mathrm{cr}(G)$.

The question is whether the inequalities are actually equalities.[1] Pach [3] called this "perhaps the most exciting open problem in the area." The only evidence for equality is an old theorem by Chojnacki [2], which was rediscovered by Tutte [6]—and the absence of any counterexamples.

Theorem 1.1 (Chojnacki, Tutte) *If* $\mathrm{ocr}(G) = 0$ *then* $\mathrm{cr}(G) = 0$.[2]

In this paper we will construct a simple example of a graph with $\mathrm{ocr}(G) \neq \mathrm{pcr}(G)$. We derive this example from studying what we call weighted maps on the annulus. Section 2 introduces the notion of weighted maps on arbitrary surfaces and gives a counterexample to $\mathrm{ocr}(M) = \mathrm{pcr}(M)$ for maps on the annulus. In Section 3 we continue the study of crossing numbers for weighted maps on the annulus, proving in particular that $\mathrm{cr}(M) \leq 3\,\mathrm{ocr}(M)$. Finally, in Section 4 we show how to translate the map counterexample from Section 2 into an infinite family of simple graphs for which $\mathrm{ocr}(G) < \mathrm{pcr}(G)$.

2 Map Crossing Numbers

A *weighted map* M is a 2-manifold S and a set $P = \{(a_1, b_1), \ldots, (a_m, b_m)\}$ of pairs of distinct points on ∂S with positive weights $w_1, \ldots, w_m$. A *realization* R of the map $M = (S, P)$ is

[1] Doug West lists the problem on his page of open problems in graph theory [7]. Dan Archdeacon even conjectured that equality holds [1].

[2] In fact they proved something slightly stronger, namely that if the independent odd crossing number is zero, then the crossing number is zero.

a set of m properly embedded arcs $\gamma_1, \ldots, \gamma_m$ in S where γ_i connects a_i and b_i. Let

$$\begin{aligned}
\mathrm{cr}(R) &= \sum_{1 \le k < \ell \le m} i(\gamma_k, \gamma_\ell) w_k w_\ell, \\
\mathrm{pcr}(R) &= \sum_{1 \le k < \ell \le m} [i(\gamma_k, \gamma_\ell) > 0] w_k w_\ell, \\
\mathrm{ocr}(R) &= \sum_{1 \le k < \ell \le m} [i(\gamma_k, \gamma_\ell) \equiv 1 \pmod 2] w_k w_\ell,
\end{aligned}$$

where $i(\gamma, \gamma')$ is the geometric intersection number of γ and γ' and $[x]$ is 1 if the condition x is true, and 0 otherwise.

We define $\mathrm{cr}(M) = \min \mathrm{cr}(R)$, where the minimum is taken over all realizations R of M. We define $\mathrm{pcr}(M)$ and $\mathrm{ocr}(M)$ analogously.

Remark 3 For every map M, $\mathrm{ocr}(M) \le \mathrm{pcr}(M) \le \mathrm{cr}(M)$.

Conjecture 1 *For every map M,* $\mathrm{cr}(M) = \mathrm{pcr}(M)$.

Lemma 2.1 *If Conjecture 1 is true then* $\mathrm{cr}(G) = \mathrm{pcr}(G)$ *for every* graph G.

Proof :
Let D be a drawing of G with minimal pair crossing number. Drill small holes at the vertices. We obtain a drawing R of a weighted map M. If Conjecture 1 is true, there exists a drawing of M with the same crossing number. Collapse the holes to vertices and obtain drawing D' of G with $\mathrm{cr}(D') \le \mathrm{pcr}(G)$. ■

We can, however, separate the odd crossing number from the crossing number for weighted maps, even in the annulus (a disk with a hole).

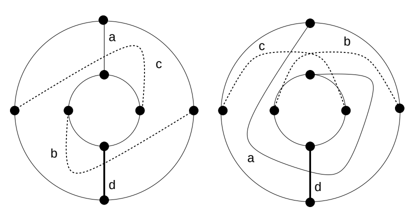

Figure 1: pcr $\neq$ ocr.

When analyzing crossing numbers of drawings on the annulus, we describe curves with respect to an initial drawing of the curve and a number of *Dehn twists*. Consider, for example, the four curves in the left part of Figure 1. Comparing them to the corresponding curves in the right part, we see that the curves labeled c and d have not changed, but the curves labeled a and b have each undergone a single clockwise twist.

Two curves are *isotopic rel boundary* if they can be obtained from each other by a continuous deformation which does not move the boundary ∂M. Isotopy rel boundary is an equivalence relation, its equivalence classes are called *isotopy classes*. An *isotopy class* on annulus is determined by a properly embedded arc connecting the endpoints, together with the number of twists performed.

Lemma 2.2 *Let $a \le b \le c \le d$ be such that $a+c \ge d$. For the weighted map M in Figure 1 we have* $\mathrm{cr}(M) = \mathrm{pcr}(M) = ac + bd$ *and* $\mathrm{ocr}(M) = bc + ad$.

Proof :
The upper bounds follow from the drawings in Figure 1, the left drawing for crossing and pair crossing number, the right drawing for odd crossing number.
Claim: $\mathrm{pcr}(M) \ge ac + bd$.
Proof of the claim: Let R be a drawing of M minimizing $\mathrm{pcr}(R)$. We can apply twists so that the thick edge d is drawn as in the left part of Figure 1. Let α, β, γ be the number of clockwise twists that are applied to arcs a, b, c in the left part of Figure 1 to obtain the drawing R. Then,

$$\mathrm{pcr}(R) = cd[\gamma \ne 0] + bd[\beta \ne -1] + ad[\alpha \ne 0] + bc[\beta \ne \gamma] + ab[\alpha \ne \beta] + ac[\alpha \ne \gamma + 1]. \quad (1)$$

If $\gamma \ne 0$ then $\mathrm{pcr}(R) \ge cd + ab$ because at least one of the last five conditions in (1) must be true; the last five terms contribute at least ab (since $d \ge c \ge b \ge a$), and the first term contributes cd. Since $d(c-b) \ge a(c-b)$, $cd + ab \ge ac + bd$, and the claim is proved in the case that $\gamma \ne 0$.

Now assume that $\gamma = 0$. Equation (1) becomes

$$\mathrm{pcr}(R) = bd[\beta \ne -1] + bc[\beta \ne 0] + ad[\alpha \ne 0] + ac[\alpha \ne 1] + ab[\alpha \ne \beta]. \quad (2)$$

If $\beta \ne -1$ then $\mathrm{pcr}(R) \ge bd + ac$ because either $\alpha \ne 0$ or $\alpha \ne 1$. Since $bd + ac \ge bc + ad$, the claim is proved in the case that $\beta \ne -1$.

This leaves us with the case that $\beta = -1$. Equation (2) becomes

$$\mathrm{pcr}(R) = bc + ad[\alpha \ne 0] + ac[\alpha \ne 1] + ab[\alpha \ne -1]. \quad (3)$$

The right-hand side of Equation (3) is minimized for $\alpha = 0$. In this case $\mathrm{pcr}(R) = bc + ac + ab \ge ac + bd$ because we assume that $a + c \ge d$. $\square$
Claim: $\mathrm{ocr}(M) \ge bc + ad$.
Proof of the claim: Let R be a drawing of M minimizing $\mathrm{ocr}(R)$. Let α, β, γ be as in the previous claim. We have

$$\mathrm{ocr}(R) = cd[\gamma]_2 + bd[\beta + 1]_2 + ad[\alpha]_2 + bc[\beta + \gamma]_2 + ab[\alpha + \beta]_2 + ac[\alpha + \gamma + 1]_2, \quad (4)$$

where $[x]_2$ is 0 if $x \equiv 0 \pmod 2$, and 1 otherwise.

If $\beta \not\equiv \gamma \pmod 2$ then the claim clearly follows unless $\gamma = 0$, $\beta = 1$, and $\alpha = 0$ (all modulo 2). In that case $\mathrm{ocr}(R) \ge bc + ab + ac \ge bc + ad$. Hence, the claim is proved if $\beta \not\equiv \gamma \pmod 2$.

Assume then that $\beta \equiv \gamma \pmod 2$. Equation (4) becomes

$$\mathrm{ocr}(R) = cd[\beta]_2 + bd[\beta + 1]_2 + ad[\alpha]_2 + ab[\alpha + \beta]_2 + ac[\alpha + \beta + 1]_2. \quad (5)$$

If $\alpha \equiv 1 \pmod 2$ then the claim clearly follows because either cd or bd contributes to the ocr. Thus we can assume $\alpha \equiv 0 \pmod 2$. Equation (5) becomes

$$\mathrm{ocr}(R) = (cd + ab)[\beta]_2 + (bd + ac)[\beta + 1]_2. \quad (6)$$

For both $\beta \equiv 0 \pmod 2$ and $\beta \equiv 1 \pmod 2$ we get $\mathrm{ocr}(R) \geq bc + ad$. □

■

We get separation of pcr and ocr for maps with small integral weights.

Corollary 2.3 *There is a weighted map M on the annulus with edges of weight $a = 1$, $b = c = 3$, and $d = 4$ for which* $\mathrm{cr}(M) = \mathrm{pcr}(M) = 15$ *and* $\mathrm{ocr}(M) = 13$.

Optimizing over the reals yields $b = c = 1$, $a = (\sqrt{3} - 1)/2$, and $d = 1 + a$, giving us the following separation of $\mathrm{pcr}(M)$ and $\mathrm{ocr}(M)$.

Corollary 2.4 *There exists a weighted map M on the annulus with* $\mathrm{ocr}(M)/\mathrm{pcr}(M) = \sqrt{3}/2$.

Conjecture 2 *For every weighted map M on the annulus,* $\mathrm{ocr}(M) \geq \frac{\sqrt{3}}{2}\,\mathrm{pcr}(M)$.

3 Crossing Numbers on the Annulus

We have seen that there can be a factor of $\sqrt{3}/2$ between $\mathrm{ocr}(M)$ and $\mathrm{cr}(M)$ for a weighted map M. In this section, we will show that the separation cannot be much larger; more precisely, for any weighted map M on the annulus,

$$\mathrm{cr}(M) \leq 3\,\mathrm{ocr}(M).$$

We first consider the special case of unit weights. Let M consist of the properly embedded arcs $\gamma_1, \ldots, \gamma_n$ with weights $w_1 = w_2 \ldots = w_n = 1$. Define the function $\mathrm{odd}(i, j, k)$ to be the odd crossing number of M restricted to γ_i, γ_j, and γ_k. Note that $\mathrm{odd}(i, j, k)$ is invariant under permuting its arguments. Consider a drawing R of M minimizing $\mathrm{ocr}(R)$, and pick two curves γ_r and γ_s that intersect an odd number of times in R. Their contribution to $\mathrm{ocr}(R)$ is 1. Now,

$$\begin{aligned}
\mathrm{ocr}(R) &= \sum_{r<s} [|\gamma_r \cap \gamma_s| \equiv 1 \pmod 2] \\
&= \frac{1}{n-2} \sum_{r<s, k \notin \{r,s\}} [|\gamma_r \cap \gamma_s| \equiv 1 \pmod 2].
\end{aligned}$$

By definition, $\mathrm{odd}(i, j, k) = 1$ implies that $|\gamma_r \cap \gamma_s| \equiv 1 \pmod 2$ for at least one pair $r, s \in \{i, j, k\}$. Therefore,

$$\begin{aligned}
& \frac{1}{n-2} \sum_{i<j<k} \mathrm{odd}(i, j, k) \\
\leq \; & \frac{1}{n-2} \sum_{r<s, k \notin \{r,s\}} [|\gamma_r \cap \gamma_s| \equiv 1 \pmod 2].
\end{aligned}$$

Let us look at the problem differently. Consider the drawing R_k which minimizes $\mathrm{cr}(R)$ under the condition that γ_k is not crossed by any other curve. Obviously, $\mathrm{cr}(R_k) \geq \mathrm{cr}(R)$.

Since curves other than γ_k are restricted to a space homeomorphic to a disk, it is easy to see that

$$\operatorname{cr}(R_k) = \sum_{i<j \text{ s.t. } k \notin \{i,j\}} \operatorname{odd}(i,j,k),$$

since three properly embedded arcs can be drawn without crossings if their odd crossing number is zero. Hence,

$$\begin{aligned}
\operatorname{cr}(R) &\leq \frac{1}{n} \sum_k \operatorname{cr}(R_k) \\
&\leq \frac{1}{n} \sum_{i<j \text{ s.t. } k \notin \{i,j\}} \operatorname{odd}(i,j,k) \\
&\leq \frac{1}{n} \cdot 3 \cdot \sum_{i<j<k} \operatorname{odd}(i,j,k).
\end{aligned}$$

Combining our estimates, we obtain

$$\operatorname{cr}(R) \leq \frac{3(n-2)}{n} \operatorname{ocr}(R) \leq 3 \operatorname{ocr}(R).$$

This proves the following lemma.

Lemma 3.1 $\operatorname{cr}(M) \leq 3 \operatorname{ocr}(M)$ *for maps M with unit weight on the annulus.*

The next step is to extend this lemma to maps with arbitrary weights.

Consider two curves γ_1, γ_2 whose endpoints on the annulus are adjacent and in the same order. In a drawing minimizing one of the crossing numbers we can always assume that the two curves are routed in parallel, following the curve that minimizes the total intersection with all curves other than γ_1 and γ_2. The same argument holds for a block of curves with adjacent endpoints in the same order. This allows us to extend Lemma 3.1 to maps with integer weights: a curve with integral weight w is replaced by w parallel duplicates.

If we scale all the weights in a map M by a factor α, all the crossing numbers will change by a factor of α^2. Hence, the case of rational weights can be reduced to integer weights. Finally, we observe that if we consider any of the crossing numbers as a function of the weights of M, this function is continuous: This is obvious for a fixed drawing of M, so it remains true if we minimize over a finite set of drawings of M. The maximum difference in the number of twists in an optimal drawing is bounded by a function of the crossing number; and thus it suffices to consider a finite set of drawings of M. We have shown:

Theorem 3.2 $\operatorname{cr}(M) \leq 3 \operatorname{ocr}(M)$ *for weighted maps M on the annulus.*

4 Separating Crossing Numbers of Graphs

We modify our annulus map example to obtain a graph G separating $\operatorname{ocr}(G)$ and $\operatorname{pcr}(G)$. Assume that the weights of the map M are integral. We can replace each pair (a_i, b_i) in the map by w_i pairs $(a_{i,1}, b_{i,1}), \ldots, (a_{i,w_i}, b_{i,w_i})$ where the $a_{i,j}$ $(b_{i,j})$ occur on ∂S in clockwise order in a small interval around of a_i (b_i). As above, the resulting map M' with unit weights will have the same crossing numbers as M.

We then replace the inner boundary of the annulus by a cycle and add a single vertex adjacent to every $a_{i,j}$ (the new edges form a *wheel* graph.) We do likewise with the outer boundary, adding a new vertex adjacent to the $b_{i,j}$.) We give all of the new edges weight W with $W > \mathrm{cr}(M)$. The weights ensure that in any optimal drawing, the wheel subgraphs will be disjoint plane graphs, such that (ignoring the weight 1 edges) there is a face bounded by the cycle on the $a_{i,j}$s and the cycle on the $b_{i,j}$s, and all weight 1 edges are drawn in that region.

Next, replace each edge of weight W by W multiple edges, obtaining a multigraph H. We can assume that in an optimal drawing of H, all multiple edges are routed in parallel, since they can follow one that minimizes intersections with other edges. Therefore, the modified wheels will still be drawn disjointly, with a single region containing all the old weight 1 edges.

Next, subdivide each of the multiple edges, obtaining a simple graph G. A drawing of H can be modified to yield a drawing of G by adding a vertex along each curve that corresponds to a subdivided edge. In fact, an optimal drawing of H can be modified in this way to obtain an optimal drawing of G.

Lemma 4.1 *For crossing number, pairwise crossing number, and odd crossing number, an optimal drawing of H can be modified by placing vertices on certain existing curves to obtain an optimal drawing of G.*

Proof :
Given an optimal drawing of H, for each curve that will be subdivided to get G, place an additional vertex near an endpoint. This yields a drawing of G for which each type of crossing number is unchanged. Thus, $\mathrm{cr}(G) \leq \mathrm{cr}(H)$, $\mathrm{pcr}(G) \leq \mathrm{pcr}(H)$, and $\mathrm{ocr}(G) \leq \mathrm{ocr}(H)$. Once we show that each of these is really an equality, we know that there is an optimal drawing of G as desired, so we are done.

Fix an optimal drawing of G. Consider any edge xz of H that is subdivided by a vertex y in G, drawn as an x, y-curve γ and a y, z-curve δ. Replacing $\gamma \cup \delta$ by a simple x, z-curve γ' contained in $\gamma \cup \delta$ will not raise the crossing number; doing likewise for all subdivided edges shows that $\mathrm{cr}(H) \leq \mathrm{cr}(G)$. Also, any curve that crosses γ' must cross either γ or δ, so the operation does not increase the pairwise crossing number. Therefore, $\mathrm{pcr}(H) \leq \mathrm{pcr}(G)$.

Instead of replacing $\gamma \cup \delta$ like that, consider each point at which γ crosses δ: We can modify both curves at the crossing to eliminate the crossing and preserve the fact that we have a simple x, y-curve and a simple y, z-curve (though which one is γ and which is δ will switch). Repeating this process will eliminate all crossings between γ and δ, and we can concatenate them to obtain a simple x, z-curve γ'. Note that any other curve crosses γ' exactly where that curve crossed $\gamma \cup \delta$. Thus, if another curve crosses γ' an odd number of times, then that curve must cross either γ or δ an odd number of times. Therefore, replacing $\gamma \cup \delta$ by γ' does not raise the odd crossing number. Doing likewise for all subdivided edges shows that $\mathrm{ocr}(H) \leq \mathrm{ocr}(G)$. ■

The lemma implies that $\mathrm{cr}(G) = \mathrm{cr}(M)$ unless an optimal drawing of G has a “flipped” wheel. Fortunately, in the flipped case the $a_{i,j}, b_{i,j}$ edges must intersect often: for fixed i, if

k of the edges have the same number of twists modulo 2, there are $\binom{k}{2} + \binom{w_i-k}{2}$ pairs with an odd number of intersections, which is at least $\binom{\lfloor w_i/2 \rfloor}{2} + \binom{\lceil w_i/2 \rceil}{2}$.

Lemma 4.2 *There exists a graph G with* $\operatorname{ocr}(G) \leq 0.937 \operatorname{pcr}(G)$.

Proof :
Choose $a = 26$, $b = 82$, $c = 194$, and $d = 220$ in Figure 2. The conditions of Lemma 2.2 are satisfied. If in a drawing D one of the wheels is flipped (and the other is not) then

$$\operatorname{ocr}(D) \geq 2\left(\binom{13}{2} + \binom{41}{2} + \binom{97}{2} + \binom{110}{2}\right) = 23098.$$

If none (or both) of the wheels are flipped in D then, by Lemma 2.2

$$\operatorname{ocr}(D) = bc + ad = 21628, \text{ and}$$

$$\operatorname{cr}(D) = \operatorname{pcr}(D) = ac + bd = 23084.$$

Hence $\operatorname{ocr}(G) = 21628$ and $\operatorname{cr}(G) = \operatorname{pcr}(G) = 23084$, which proves the lemma. ■

With a little more work, we can get a better separation. Let us go back to the weighted map counterexample from Corollary 2.4 (also see Figure 1 again). For any given $m \geq 1$, let $a = \lfloor m\frac{\sqrt{3}-1}{2} \rfloor$, $b = c = m$, and $d = \lfloor m\frac{\sqrt{3}+1}{2} \rfloor$. Then Lemma 2.2 applies, and we have $\operatorname{cr}(M) = \operatorname{pcr}(M) = m\lfloor m\frac{\sqrt{3}-1}{2} \rfloor + m\lfloor m\frac{\sqrt{3}+1}{2} \rfloor > \sqrt{3}m^2 - 2$, while $\operatorname{ocr}(M) \leq (3/2)m^2$. Hence, we can build maps with integer weights such that $\operatorname{ocr}(M) \leq (\frac{\sqrt{3}}{2} + o(1)) \operatorname{pcr}(M)$. We have seen earlier how to turn such a map into a graph G, which suffices unless the optimum $\operatorname{pcr}(G)$ occurs for a drawing with a flipped wheel. Consider the case then that the inner wheel flips over. That is, we are looking at the map M' drawn in Figure 2.

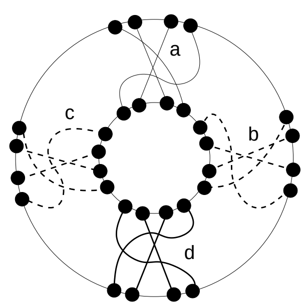

Figure 2: The inside flipped.

We will show that $\operatorname{ocr}(M') \sim 2m^2$. Since $\operatorname{cr}(M) = m\lfloor m\frac{\sqrt{3}-1}{2} \rfloor + m\lfloor m\frac{\sqrt{3}+1}{2} \rfloor \leq \sqrt{3}m^2$, this means that G will be drawn like M rather than M' for large enough m, whether our goal is to optimize cr, pcr, or ocr. In other words, the inner wheel does not flip.

Theorem 4.3 *There are graphs G with* $\operatorname{ocr}(G) \leq (\frac{\sqrt{3}}{2} + o(1)) \operatorname{pcr}(G)$.

To finish the proof, it suffices to show that $\mathrm{ocr}(M') \sim 2m^2$. Since we are minimizing $\mathrm{ocr}(M')$, any curve in the drawing has just two choices: it is drawn as in Figure 2, or it has one additional twist (two twists make the same contribution to $\mathrm{ocr}(M')$ as no twist).

To simplify our formulas, let $n_1 := a$, $n_2 := b$, $n_3 := c$, and $n_4 := d$. A glance at Figure 2 will convince the reader that

$$\mathrm{ocr}(M') = \min_{0 \le k_i \le n_i} \sum_i \binom{k_i}{2} + \sum_i \binom{n_i - k_i}{2} + \sum_{i \ne j} k_i(n_j - k_j).$$

Let $x_1 := \frac{\sqrt{3}-1}{2}$, $x_2 := x_3 := 1$, and $x_4 := \frac{\sqrt{3}+1}{2}$, and observe that $|n_i - mx_i| < 1$ for $1 \le i \le 4$. Define

$$g = \min_{y_i + z_i = x_i} \sum_i \binom{my_i}{2} + \sum_i \binom{mz_i}{2} + \sum_{i \ne j} my_i mz_j$$

(where $\binom{a}{2} = a(a-1)/2$). Observe that g is close to $\mathrm{ocr}(M')$:

$$\begin{aligned}
|g - \mathrm{ocr}(M')| &< \sum_i my_i + \sum_i mz_i + \sum_{i \ne j}(my_i + mz_j - 1) \\
&< m \sum_{i,j}(y_i + z_j) \\
&= 4m \sum_i (y_i + z_i) \\
&= 4m(\sqrt{3} + 2).
\end{aligned}$$

Now let

$$f = \sum_i y_i^2/2 + \sum_i z_i^2/2 + \sum_{i \ne j} y_i z_j,$$

with the y_i, z_i optimized according to g. Note that $m^2 f - g = \sum_i my_i/2 + \sum_i mz_i/2 = m(\sqrt{3}+2)/2$. Presently, we will show $f \ge 2$. Assuming this for the moment, we obtain $\mathrm{ocr}(M') > g - 4m(\sqrt{3}+2) \ge 2m^2(1 - 2.25(\sqrt{3}+2)/m)$.

Our motive for showing $\mathrm{ocr}(M') \sim 2m^2$ was to show that $\mathrm{ocr}(M') > \mathrm{cr}(M)$. Since $\sqrt{3}m^2 \ge \mathrm{cr}(M)$, it suffices to have $2m^2(1 - 2.25(\sqrt{3}+2)/m) \ge \sqrt{3}m^2$, or $m \ge \frac{2.25(\sqrt{3}+2)}{1-\sqrt{3}/2}$, which is less than 63.

We are left with the proof of the estimate.

Lemma 4.4 *For each i, let y_i, z_i be nonnegative real numbers such that $x_i = y_i + z_i$, where $x_1 = \frac{\sqrt{3}-1}{2}$, $x_2 = x_3 = 1$, and $x_4 = \frac{\sqrt{3}+1}{2}$. Then the value*

$$f = \sum_i y_i^2/2 + \sum_i z_i^2/2 + \sum_{i \ne j} y_i z_j$$

is at least 2.

Proof :

Suppose that x_1, x_2, x_3, x_4 take the values $L = \frac{\sqrt{3}-1}{2}, 1, 1, H = \frac{\sqrt{3}+1}{2}$, not necessarily in that order. For each i, let y_i, z_i be nonnegative real numbers such that $x_i = y_i + z_i$. Let us minimize $f = \sum_i y_i^2/2 + \sum_i z_i^2/2 + \sum_{i \neq j} y_i z_j$.

Let $d_i = z_i - y_i$. Note that $\frac{\partial f}{\partial y_i} = -d_i + \sum_{j \neq i} d_j$.

Therefore, for any extrema not on the boundary, $d_i = \sum_{j \neq i} d_j$ for all i. It follows that every $d_i = 0$. Hence $y_i = z_i = x_i/2$ for each i, in which case we obtain $f = 7/4 + \sqrt{3}$.

We can now restrict our search to the boundary; thus we may assume that y_1 or z_1 is zero. By symmetry, $y_1 = 0$. In the restricted solution space, any extrema not on the boundary must satisfy (via partial derivatives) $d_i = \sum_{j \neq i} d_j$ for $i = 2, 3, 4$. This yields $-d_1 = d_2 = d_3 = d_4$. Since $-d_1 = z_1 - y_1 = x_1$ and each $|d_i| \leq x_i$, $x_1 = L$. It follows that for each $i \neq 1$, $y_i = (x_i - L)/2$ and $z_i = (x_i + L)/2$, whence we obtain $f = 11/4 + \sqrt{3}/2$.

Otherwise, we may assume that y_i or z_i is zero for $i = 1, 2$. Again, for extrema not on the boundary of the restricted solution space, $d_i = \sum_{j \neq i} d_j$ for $i = 3, 4$. This yields $d_3 = d_4$ and $d_1 + d_2 = 0$. Hence we may assume $y_1 = 0, z_1 = 1, y_2 = 1, z_2 = 0$. Let $w = d_3$. Then for $i = 3, 4$, $y_i = (x_i - w)/2$ and $z_i = (x_i + w)/2$. Then $\frac{\partial f}{\partial w} = -d_3/2 - d_4/2$, which we set to zero, yielding $d_3 = 0 = d_4$. Then $y_3 = z_3$ and $y_4 = z_4$, which take the values $L/2$ and $H/2$. Then $f = 11/4 + \sqrt{3}$.

Otherwise, we may assume that y_i or z_i is zero for $i = 1, 2, 3$. Then either $d_4 = d_1 + d_2 + d_3$, or y_4 or z_4 is zero (a *corner* of the entire solution space). First consider the non-corner case(s):

If $x_4 = L$, then we may assume that $x_1 = x_2 = 1, x_3 = H$. Since $|d_i| = x_i$ for $i = 1, 2, 3$, $|d_1 + d_2|$ is 0 or 2, which implies that $|d_1 + d_2 + d_3|$ is H, $2 + H$, or $2 - H$. Each is greater than $L = |d_4|$, a contradiction.

If $x_4 = H$, then we may assume that $x_1 = x_2 = 1, x_3 = L$. If $|d_1 + d_2| = 2$ then $|d_1 + d_2 + d_3| \geq 2 - L > H$, a contradiction. We may assume that $y_1 = 0, z_1 = 1, y_2 = 1, z_2 = 0, y_3 = 0, z_3 = L$, which yields $y_4 = (H - L)/2$, $z_4 = (H + L)/2$, and $f = 11/4 + \sqrt{3}$.

If $x_4 = 1$, then we may assume that $x_1 = H, x_2 = 1, x_3 = L$. Then d_1 and d_2 must have opposite signs, since $H + 1 - L > 1$. We may assume that $y_1 = 0, z_1 = H, y_2 = 1, z_2 = 0$. If $y_3 = L$, then $d_4 = H - 1 - L = 0$, so $y_4 = z_4 = 1/2$. If $z_3 = L$, then $d_4 = H - 1 + L = \sqrt{3} - 1$, which yields $y_4 = 1 - \sqrt{3}/2$, $z_4 = \sqrt{3}/2$. These cases yield $f = 13/4 + \sqrt{3}$ and $f = 5/4 + 2\sqrt{3}$, respectively.

Finally, we consider the corners: Suppose that for all i, y_i or z_i is zero. We may assume that $y_1 = 0, z_1 = L$, $x_2 = x_3 = 1$, and (y_4, z_4) is either $(0, H)$ or $(H, 0)$. Without loss of generality, (y_2, y_3) is either $(0, 0)$, $(1, 0)$ or $(1, 1)$. For these 6 possibilities f is 2, $3 + \sqrt{3}$, $2 + 2\sqrt{3}$, or (thrice) $3/2 + \sqrt{3}$.

Of all these possibilities, 2 is the smallest, hence $f \geq 2$. ■

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