Strong Reductions and Immunity for Exponential Time

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Abstract

This paper investigates the relation between immunity and hardness in exponential time. The idea that these concepts are related originated in computability theory where it led to Post's program, and it has been continued successfully in complexity theory [9, 13, 20]. We study three notions of immunity for exponential time. An infinite set A is called

- **EXP**-immune, if it does not contain an infinite subset in **EXP**;
- **EXP**-hyperimmune, if for every infinite sparse set $B \in \mathbf{EXP}$ and every polynomial p there is an $x \in B$ such that $\{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$ is disjoint from A;
- **EXP**-avoiding, if the intersection $A \cap B$ is finite for every sparse set $B \in \mathbf{EXP}$.

EXP-avoiding sets are always **EXP**-hyperimmune and **EXP**-hyperimmune sets are always **EXP**-immune but not vice versa. We analyze with respect to which polynomial-time reducibilities these sets can be hard for **EXP**. **EXP**-immune sets cannot be conjunctively hard for **EXP**although they can be disjunctively hard. **EXP**-hyperimmune sets cannot be conjunctively or disjunctively hard for **EXP**, but there is a relativized world in which there is an **EXP**-avoiding set which is hard with respect to positive truth-table reducibility. Furthermore, in every relativized world there is some **EXP**-avoiding set which is Turing-hard for **EXP**.

1 From Post's Program to Complexity Theory

Concepts of immunity have a long tradition in computability theory beginning with the famous paper of Post [11, 18] which introduced simple sets and showed that they are not hard in the sense that the halting problem cannot be many-one reduced to a simple set. In fact, no set without an infinite computable subset can be many-one hard for the halting problem. These sets are called *immune* and the present paper extends the study of resource bounded versions of

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this notion. Post also considered the more restrictive notions of hyperimmune and hypersimple sets which are not truth-table hard for the halting problem. Post's program to find a Turing-incomplete set defined by abstract properties more restrictive than being simple was completed in the 1970s when Dëgtev and Marchenkov proved that η-maximal semirecursive simple sets exist for a suitable positive equivalence-relation η and that such sets neither have the Turing-degree of the empty set nor of the halting problem, but are intermediate [17, Section III.5]. Furthermore, Harrington and Soare [12] found a condition which is definable in the lattice defined by set-inclusion of the computably (recursively) enumerable sets and which enforces that these sets have an intermediate Turing degree.

Within complexity theory, Berman and Hartmanis [8] started the search for structural properties which imply that a set is not hard for the classes **NP**, **PSPACE**, or **EXP**. Hartmanis, Li and Yesha [13] studied whether **NP**-simple sets can be complete for **NP** where a set A is **NP**-simple iff $A \in \mathbf{NP}$ and A is co-infinite and no infinite set $B \in \mathbf{NP}$ is disjoint to A. They showed that an **NP**-simple set cannot be many-one hard, unless every problem in **NP** can be decided in subexponential time. Agrawal communicated to the first author, that under the assumption that **P** and **NP** are not equal, no **NP**-simple set A can be complete for **NP** with respect to honest bounded truth-table reducibility. Fenner and Schaefer [20] showed that there no **NP**-hyperimmune set is hard for **NP** with respect to honest Turing reducibility if there is a set $A \in \mathbf{NP} \setminus \mathbf{P}$ which has unique witnesses, that is, $A = \{x : (\exists y \in \{0,1\}^{p(|x|)})[(x,y) \in B]\}$ where p is a polynomial, $B \in \mathbf{P}$ and there is, for every x, at most one $y \in \{0,1\}^{p(|x|)}$ with $(x,y) \in B$. Furthermore, under various assumptions similar results were shown with respect to arbitrary, not necessarily honest polynomial time reducibilities.

The class **EXP** is much more well-behaved than **NP**. It is different from **P** in all relativized worlds and therefore contains difficult sets. Furthermore, one can build sets in **EXP** by doing a polynomially length-bounded search within other sets in **EXP**. For example, one can construct for any set in **EXP** a sparse but infinite subset which is still in **EXP**. The study of immunity notions for **EXP** often yields results like this which are true for all relativized worlds and do not depend on any unproven assumptions such as the non-collapse of the polynomial hierarchy or some **NP**-complete problem not being computable in subexponential time.

In the present paper, we investigate three immunity-notions for **EXP**. Namely, an infinite set A is

- **EXP**-immune, if it does not contain an infinite subset in **EXP**;
- **EXP**-hyperimmune, if for every infinite sparse set $B \in \mathbf{EXP}$ and every polynomial p there is an $x \in B$ such that $\{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$ is disjoint from A,
- **EXP**-avoiding, if the intersection $A \cap B$ is finite for every sparse set $B \in \mathbf{EXP}$.

Every **EXP**-avoiding set is **EXP**-hyperimmune and every **EXP**-hyperimmune set is **EXP**-immune. Note that the condition of B being sparse is necessary in the definition of **EXP**-avoiding sets, since every infinite set has an infinite intersection with a set in **P** namely $\{0,1\}^*$.

In this paper we investigate whether sets immune in one of the senses above can be hard for **EXP** for different types of reducibilities. This continues similar research by Fenner and Schaefer [20] on **NP**-simple, **NEXP**-simple and **NP**-immune sets.

2 Basic Definitions and Theorems

The complexity classes **P**, **Q** and **EXP** denote the sets of languages for which there is a constant c such that their characteristic function is computable with time bound n^c, $n^{\log^c(n)}$ and 2^{n^c}, respectively, where n is the length of the corresponding input (viewed upon as a binary string; actually n has to be set to be 3 if the string has length 0, 1 or 2). **NP** is the class corresponding to **P** which permits non-deterministic computations, that is, $A \in$ **NP** iff there is a function M and a constant c such that, for all $x \in A$, some computation of M with input x halts in time n^c and, for all $x \notin A$, no computation with input x halts, whatever time the computation needs. Note that non-determinism permits M to have different computations on the same input, this is an essential part of the definition of **NP**.

The following definition of an **EXP**-immune set is analogous to those of **P**-immune, **NP**-immune and **PSPACE**-immune sets found in the literature [6].

Definition 2.1 *A set A is* **EXP***-immune iff A is infinite and it does not contain an infinite subset in* **EXP***.*

We observed earlier that every infinite set in **EXP** has an infinite sparse subset in **EXP**, it follows that a set A is **EXP**-immune iff A does not have an infinite sparse subset in **EXP**. This property is strengthened in the following definition of **EXP**-avoiding sets.

Definition 2.2 *A set A is* **EXP***-avoiding iff A is infinite and the intersection of A with any sparse set in* **EXP** *is finite.*

The next definition is obtained by adapting the notion of **NP**-hyperimmune from Fenner and Schaefer [20] to exponential time. In the definition we use the notation "$f\{x\}$" to indicate that the output of f is not a string but a set of strings which, of course, could be coded as a string again.

Definition 2.3 *Call a (partial) function f an* **EXP**-array*, if f is computable in* **EXP***, it has infinite domain and there is a polynomial p such that the cardinality of $f\{y\}$ is at most $p(|y|)$ for all y and $p^{-1}(|y|) \leq |z| \leq p(|y|)$ for all $z \in f\{y\}$, where $p^{-1}(n) = \min\{m : p(m) \geq n\}$. (The final condition assures that f is honest with respect to every element it outputs.)*

A set A is **EXP***-hyperimmune iff for all* **EXP***-arrays f there is an x in the domain of f such that $f\{x\}$ and A are disjoint.*

The following theorem shows that the definition of **EXP**-hyperimmunity given in Definition 2.3 is equivalent to the one given in the introduction.

Theorem 2.4 *A set A is* **EXP***-hyperimmune iff, for every infinite* **EXP***-sparse set B and every polynomial p, there is an $x \in B$ such that $\{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$ is disjoint from A.*

Proof. Assume that A is **EXP**-hyperimmune, B is an infinite **EXP**-sparse set and p is an increasing polynomial with $p(n) > n$ for all n. Then define the function f by taking B as the domain of f and

$$f\{x\} = \{y \in B : p^{-1}(|x|) \leq |y| \leq p(|x|)\}$$

Clearly, f is computable in exponential time. As B is sparse, there is an increasing polynomial q such that $B \cap \{0,1\}^n$ contains less than $q(n)$ elements for every n. It follows that $f\{x\}$ has at most $q(p(|x|)) \cdot p(|x|)$ many elements, and so f is an **EXP**-array. Hence, there is an x such that $f\{x\}$ is disjoint from A and thus the condition given in the theorem is satisfied by every **EXP**-hyperimmune set.

For the converse direction, consider any set A satisfying the condition of the theorem and let f be an **EXP**-array with respect to an increasing polynomial q where $q(n) > n$ for all n. Let $p(n) = q(q(n)) + 1$ and define a set $B = B_0 \cup B_1 \cup \ldots$ in stages as follows.

Stage 0**:** Let $B_0 = \emptyset$ and $l_0 = 0$.

Stage $s+1$**:** Check whether there is a string x in the domain of f such that $l_s \leq |x| \leq p(l_s)$ and all elements $y \in f\{x\}$ have at least length l_s.

(I) If so, let $B_{s+1} = B_s \cup f\{x\}$ for the first such x found and let $l_{s+1} = p(|x|) + 1$.

(II) If not, let $B_{s+1} = B_s$ and $l_{s+1} = l_s + 1$.

For the verification, one sees easily that the set $B = \cup_s B_s$ has at length n at most $p(p(n))$ many elements and is thus sparse. Let r be a polynomial such that f is computable in time $2^{r(|x|)}$ for all x in its domain. Then one can do any step s with $l_s \leq n$ in time $O(2^{r(p(n))} \cdot 2^{p(n)})$ and thus compute the characteristic function of B in time $O(n \cdot 2^{r(p(n))} \cdot 2^{p(n)})$. So B is in **EXP**.

Now there is a $y \in B$ such that $B \cap A$ does not contain any element of length n with $p^{-1}(|y|) \leq n \leq p(|y|)$. This y came into B as an element of some $f\{x\}$ and therefore $q^{-1}(|x|) \leq |y| \leq q(|x|)$. This condition is equivalent to $q^{-1}(|y|) \leq |x| \leq q(|y|)$. As $p(m) > q(q(m))$ for all m, it follows that every element of $f\{x\}$ has a length n with $p^{-1}(|y|) \leq n \leq p(|y|)$ and thus $f\{x\}$ is disjoint to A. So A is **EXP**-hyperimmune. □

If the intersection of A with an infinite sparse set B is finite, then the condition in the above theorem is clearly satisfied for every polynomial. Thus one obtains the following corollary.

Corollary 2.5 *Every* **EXP**-*avoiding set is* **EXP**-*hyperimmune.*

A reducibility is an algorithm to compute a set A relative to a set B. B is often called an *oracle*. In this paper we only consider polynomial time algorithms for reducibilities. Different types of reducibilities are obtained by restricting the algorithm, and its access to B; we include a partial list.

Turing reducibility. A is Turing reducible to B iff there is a polynomial p such that some Turing machine M computes $A(x)$ in time $p(|x|)$ with queries to B. We write $A(x) = M^B(x)$. Due to time constraints there are at most $p(|x|)$ queries and every y in a query satisfies $|y| \leq p(|x|)$. If it also satisfies $|y| \geq p^{-1}(|x|)$ for every y queried by M, then the reduction is called *honest*.

Truth-table reducibility (tt-reducibility). A is tt-reducible to B iff there is a Turing machine M and a polynomial-time computable function f such that $f\{x\}$ is a set of polynomially many strings and M computes A relative to B only querying elements of the set $f\{x\}$.

Positive reducibility (ptt-reducibility). A is ptt-reducible to B iff A is tt-reducible to B via a function f and machine M as defined previously and, moreover, for any two sets $D \subseteq E$ and any input x we have that $M^D(x) \leq M^E(x) \in \{0,1\}$.

Conjunctive reducibility (c-reducibility). A is c-reducible to B iff there is a polynomial time computable function f such that $x \in A$ iff $f\{x\} \subseteq B$.

Disjunctive reducibility (d-reducibility). A is d-reducible to B iff there is a polynomial time computable function f such that $x \in A$ iff $f\{x\}$ meets B, that is, $f\{x\} \cap A \neq \emptyset$.

Parity-reducibility. A is parity-reducible to B iff there is a polynomial time computable function f such that $x \in A$ iff the cardinality of the intersection $f\{x\} \cap B$ is odd. In computability theory, parity-reducibility is often called linear reducibility [17, page 269].

Many-one reducibility (m-reducibility). A is m-reducible to B iff there is a polynomial time computable function f such that $x \in A$ iff $f(x) \in B$.

The function f in the definitions of positive, conjunctive, disjunctive and parity-reducibility has as output not a single string but a set of strings. This is indicated by writing $f\{x\}$ instead of $f(x)$. Furthermore, a $g(n)$-r-reducibility is a reducibility where on input of length n one can ask at most $g(n)$ questions. For example, a $\log(n)$-tt-reduction requires that the cardinality of the set $f\{x\}$ is always at most $\log(|x|)$.

In the following, reducibilities are also used to define a generalization of the notion of classes using advice. The best-known related concept is the class **P/poly** which is the class of all sets that can be Turing-reduced to polynomial-sized advice.

Definition 2.6 *We say* **EXP** *is* compressible via a reducibility r *iff for every set E in* **EXP** *there is an r-reduction M and for infinitely many lengths $n > 0$ there is a set $A_n \subseteq \{0,1\}^{<n}$, called the* advice *such that M r-reduces E to A_n on the domain $\{0,1\}^n$.*

Theorem 2.7 **EXP** *is incompressible via any of the following reducibilities: conjunctive reducibility, disjunctive reducibility, parity-reducibility. This result relativizes.*

Note that **EXP** $\subseteq$ **P/poly** iff there is a tally Turing-complete set for **EXP**. Wilson [22] constructed a relativized world in which **EXP** $\subseteq$ **P/poly**

Hence, we cannot expect to improve the statement of the theorem to Turing reductions without making further assumptions.

Proof. Recall that the reducibilities above compute on input x some set $f\{x\}$ such that $x \in E$ iff $f\{x\}$ intersects A_n in case of disjunctive reducibility, $f\{x\}$ is a subset of A_n in the case of conjunctive reducibility, $f\{x\}$ has an odd number of elements in common with A_n in the case of parity-reducibility.

Disjunctive Reducibility. Given a reducibility $f = \varphi_e$ which—without loss of generality—is computable in time 2^e on input of length e one defines a partial function g from $\{0,1\}^e$ to $\{0,1\}^{<e}$ which on input x outputs some string z iff this z (and perhaps some other ones) is in the set $f\{x\}$ but not in any set $f\{y\}$ with $y \in \{0,1\}^e \setminus \{x\}$. Therefore, whenever $g(x)$ and $g(y)$ are defined, then $g(x) \neq g(y)$. As there are 2^e strings of length e but only $2^e - 1$ strings of strictly shorter length, there is a string $x \in \{0,1\}^e$ such that $g(x)$ is undefined. For each e, let x_e as the lexicographically first string in $\{0,1\}^e$ where g is undefined and let $E = \{x_0, x_1, \ldots\}$ be the set of all these x_e.

To see that E is in **EXP** note that g is based on the function $f = \varphi_e$ which is computable in time 2^e. We initialize an array of length 2^e with entry 0 for every $x \in \{0,1\}^e$. For every z we can check in time 4^e whether $z \in \varphi_e\{x\}$ for exactly one $x \in \{0,1\}^e$ and if so, set the corresponding entry to 1. Repeating this for all z allows us to compute x_e as the lexicographic first string in $\{0,1\}^e$ whose entry is still 0 in at most 8^e steps.

The function $f = \varphi_e$ does not compute $E \cap \{0,1\}^e$ using any advice A_e, because $x_e \in E$ implies that A_e intersects $\varphi_e\{x_e\}$ at some element z and, since $g(x_e)$ is undefined, there is some $y \in \{0,1\}^e \setminus \{x_e\}$ with $z \in \varphi_e\{y\}$. It would follow $y \in E$ which contradicts to x_e being the only element of E of length e.

Conjunctive Reducibility. The proof is analogous to the one for disjunctive reducibility. We can take the complement $\overline{E}$ of the previously constructed set E and use the fact that φ_e reduces E disjunctively to advice A_e iff φ_e reduces $\overline{E}$ conjunctively to $\overline{A_e}$.

Parity-Reducibility. For the case of parity-reducibility we use the fact that every $f = \varphi_e$ defines a linear mapping from the $2^n - 1$ dimensional Boolean vector space of the subsets of $\{0,1\}^{<n}$ into the vector space of the subsets of $\{0,1\}^n$ for each n. As the space of all characteristic functions on $\{0,1\}^n$ is 2^n-dimensional, there is some possible characteristic function on $\{0,1\}^n$ which is not in the linear closure of the images $\varphi_e^{-1}(\{z\})$ where $|z| < n$. Hence, for every length $n = e$, we can determine in exponential time the characteristic function on all strings of length e in such a way that it does not coincide with any possible image $\varphi_e^{-1}(A_e)$ for any $A_e \subseteq \{0,1\}^{<n}$. Thus we get a set $E \in$ **EXP** which is incompressible via parity-reducibility. □

3 Immunity and Hardness for EXP

The goal of this chapter is to investigate for which reducibilities r there are **EXP**-immune, **EXP**-hyperimmune, and **EXP**-avoiding sets which are r-hard for **EXP**.

Theorem 3.1 *No c-hard set for* **EXP** *is* **EXP***-immune.*

Proof. From Theorem 2.7 it follows that there is a set $\{x_0, x_1, \ldots\} \in$ **EXP** whose complement E is incompressible via conjunctive reducibility. Assume by way of contradiction that there is a conjunctive reduction f from E to some **EXP**-immune set A and let $F = \{x : f\{x\}$ contains some string z with $|z| \geq |x|\}$. If $E \cap F$ is infinite then the set

$$U = \{z : (\exists x \in E \cap F)\,[|x| \leq |z| \wedge z \in f\{x\}]\}$$

is also infinite and it is a subset of A, since $f\{x\} \subseteq A$ for every $x \in E$. This contradicts A being **EXP**-immune. So $E \cap F$ is finite and we can modify f to obtain the following function

g also computable in polynomial time. Without loss of generality let $\lambda \in A$. Now g is defined by taking the first of the following cases which applies.

$$g\{x\} = \begin{cases} f\{x\} & \text{if } f\{x\} \subseteq \{0,1\}^{<|x|}; \\ \{\lambda\} & \text{if } x \in E \cap F; \\ \emptyset & \text{otherwise.} \end{cases}$$

This g would then witness that E can be compressed using a conjunctive reduction, contradicting the choice of E. Thus E cannot be c-reduced to an **EXP**-immune set. □

Buhrman [9] showed that there are d-complete sets in **EXP** which are **P**-immune. This result also holds for higher levels.

Fact 3.2 (Buhrman [9]) *Let* **DEXP** *denote the class of all sets which are computable in double exponential time, that is, in time* $2^{2^{p(n)}}$ *for some polynomial* p*. For a given* **DEXP***-complete set* A *one can construct an* **EXP***-immune set* $B \in$ **DEXP** *such that* $x \in A \Leftrightarrow x0 \in B \vee x1 \in B$*.* B *is clearly d-complete for* **DEXP** *and d-hard for* **EXP**.

In contrast to the existence of **EXP**-immune sets which are d-hard for **EXP**, the next result shows that d-hard sets for **EXP** cannot be **EXP**-hyperimmune.

Theorem 3.3 *No d-hard set for* **EXP** *is* **EXP***-hyperimmune.*

Proof. Let $E = \{x_0, x_1, \ldots\}$ be the set constructed in Theorem 2.7 which is incompressible with regard to disjunctive reductions. Assume that $E \leq_d A$ for an hyperimmune set A via a reduction g, that is, $x \in E$ iff $g\{x\} \cap A \neq \emptyset$. By the padding-lemma there is an infinite polynomial-time computable set U of indices of g. For every $e \in U$ we can compute for every $z \in f_e\{x_e\}$ with $|z| < |x_e|$ one input $y \in \{0,1\}^e \setminus \{x_e\}$ such that $z \in f_e\{y\}$ (this is possible by the definition of x_e). This gives a polynomial-sized subset F_e of $\{0,1\}^e$. We define

$$h\{x_e\} = \{z : (\exists y \in F_e \cup \{x_e\}) \, [|z| \geq e \wedge z \in f_e\{y\}]\}$$

where $h\{y\}$ is undefined whenever $y \notin E$ or $|y| \notin U$. As U is infinite, h has an infinite domain and there is an e such that $h\{x_e\} \cap A = \emptyset$. Now f_e disjunctively reduces the elements of $F_e \cup \{x_e\}$ to $A^{<e}$, which contradicts the construction in Theorem 2.7. □

Theorem 3.3 leads to the question whether **EXP**-hyperimmune sets can be tt-hard for **EXP**. This question is still open, but we can show that **EXP**-hyperimmune sets cannot be hard for **EXP** under n^α-tt reductions where $\alpha < 1$. As often happens in these cases, the problem changes character at $\alpha = 1$. There are several preliminary results for Theorem 3.4. Buhrman [10] proved the related result that no **NEXP**-simple set is btt-hard for **EXP** where a **NEXP**-simple set is a co-infinite set $A \in$ **NEXP** such that no infinite set in **NEXP** is disjoint to A. Later, Schaefer [19] showed that no $\alpha \log(n)$-tt-hard set for **EXP** is **EXP**-hyperimmune, where α can be any constant; the proof of this result implied Buhrman's result. It also showed that **NP**-simple sets cannot be btt-hard for **EXP**. Buhrman also showed that no **EXP**-hyperimmune set can be n^α-tt-hard for **EXP** where $\alpha < \frac{1}{3}$. Theorem 3.4 improves this bound by showing the result for all $\alpha < 1$.

Theorem 3.4 *An* **EXP***-hyperimmune set cannot be* n^α*-tt-hard for* **EXP** *where* $\alpha < 1$.

Proof. This result is obtained by constructing a set $E \in$ **EXP** which cannot be n^α-tt-reduced to any **EXP**-hyperimmune set A for any $\alpha < 1$. Fix an **EXP**-hyperimmune set A.

At each level $\{0,1\}^n$ with $n = \langle i,j,k\rangle$ and $k \geq 3$, the set E contains at most one element x which is picked to diagonalize against $g_{i,k}$, the i-th $n^{(k-2)/k}$-tt-reduction to A. We can assume without loss of generality, that each of these reductions does not exceed the computation time 2^n which is clearly an upper bound for the polynomial time required by the i-th reduction.

For every $x \in \{0,1\}^n$ g_n determines a particular truth-table. We modify this truth-table as follows: queries of length $n^{1/k}$ or more are answered 0. Assign this modified truth-table to x. The number of these modified truth-tables is at most $p^m \cdot 2^m$ where m is the maximum number of queries, and p is the number of queries of length less than $n^{1/k}$. Thus $p \leq 2^{n^{1/k}}$, $m = n^{(k-2)/k}$ and $p^m \leq 2^{n^{(k-1)/k}}$. It follows that, for fixed i,k, almost all parameters j satisfy that $2^n > 2^{n^{(k-1)/k}+n^{(k-2)/k}}$, meaning that there are more strings x of length n than modified truth-tables. If this is the case, then there are two different strings $x_n, y_n \in \{0,1\}^n$ such that the modified truth-tables assigned to these strings are the same. Furthermore, given 0^n as an input we can find two such strings (if they exist) in exponential time.

Hence we can define the set $E = \{x_n\text{: if } x_n, y_n \text{ exist}\}$ in **EXP**. For $z = 0^n$ we define a set $f_{i,k}\{z\}$ if x_n and y_n exist and $n = \langle i,j,k\rangle$ for some j as follows: $f_{i,k}\{0^n\}$ contains all those strings of length $n^{1/k}$ or more which are queried by the i-th $n^{(k-2)/k}$-tt-reduction on input x_n or y_n. Note that every function $f_{i,k}$ is honest, since on input z it produces strings of length at least $|z|^{1/k}$.

Assuming that A is **EXP**-hyperimmune and that the i-th $n^{(k-2)/k}$-tt-reduction reduces E to A, the domain of $f_{j,k}$ is infinite and there is a string z in the domain of $f_{j,k}$ where $A \cap f_{i,k}\{z\}$ is empty. The length n of z is equal to $\langle i,j,k\rangle$ for some j, and x_n, y_n exist. For both numbers, the modified and original truth-tables with respect to $f_{i,k}$ are the same, since the queries above $n^{1/k}$ are made to non-elements of A. It follows that $f_{i,k}$ assigns the same value to both x_n and y_n, although $x_n \in E$ and $y_n \notin E$, implying that $g_{i,k}$ does not reduce E to A.

It follows that A is not hard for **EXP** with respect to truth-table reductions making at most n^α queries where $\alpha < \frac{k-2}{k}$ for some k, that is, $\alpha < 1$. □

Corollary 3.5 *An* **EXP***-hyperimmune set cannot be* $\alpha\log(n)$*-Turing-hard for* **EXP** *where* $\alpha < 1$.

This bound is almost optimal as the next result shows that there is even an **EXP**-avoiding set which is $\log^2(n)$-Turing-hard for **EXP**.

Theorem 3.6 *There is an* **EXP***-avoiding set which is* $\log^2(n)$*-Turing-hard for* **EXP**.

Proof. Let E be an many-one-complete set for **EXP** and let $\{y_0, y_1, \ldots\}$ be a set of strings such that y_n has length n and has maximal Kolmogorov complexity with respect to the time-bound 2^{2^n}. Let F be the set with the characteristic function $y_0y_1y_2\ldots$ with respect to length-lexicographic ordering.

The function f mapping n to the first integer number larger than $\frac{1}{2}\cdot\log^2(n)$ is polynomial-time computable and increasing. Now consider $A = \{xy_{f(|x|)} : x \in E \oplus F\}$. The Kolmogorov complexity with respect to the time bound $2^{2^{f(|x|)}}$ of every string $xy_{f(|x|)}$ is at least $f(|x|) - c$ where c is some constant independent of $|x|$. It follows that the intersection of A and any given sparse set in **EXP** is finite. On the other hand, A is infinite, and so A is **EXP**-avoiding and **EXP**-hyperimmune.

A Turing-reduction from $E \oplus F$ to A first computes for input x the value $f(|x|)$ and then reconstructs $y_{f(|x|)}$ with queries to places $x'y_{f(|x'|)}$ where x' are the positions in the F-part of the join $E \oplus F$ coding the bits of $y_{f(|x|)}$. The bits of the $y_{f(|x'|)}$ itself have to be reconstructed first by going down a further iteration and to get these, one has even to get one more step down. But then one can compute these very short strings as every string y_n is computable in time 2^{2^n+n}. It is easy to see that the number of queries needed by the whole procedure is in $f(|x|)+O(\log(|x|))$. One remaining query to establish whether $x \in E \oplus F$ is added by querying whether $xy_{f(|x|)}$ is in A. □

It is unknown what happens in the case of a $\log(n)$ queries at a Turing-reduction or, in general, at any truth-table reduction. The next result shows that there is at least a relativized world where there is a truth-table hard set for **EXP**. Remember that there is a relativized world in which **EXP** $\subseteq$ **P/poly** [22].

Theorem 3.7 *In the relativized world where* **EXP** $\subseteq$ **P/poly***, some* **EXP***-avoiding set is hard for* **EXP** *with respect to positive truth-table reducibility.*

Proof. If **EXP** $\subseteq$ **P/poly** then there is a tally Turing-hard set E for **EXP**. Assuming that $0^{2n} \in E \Leftrightarrow 0^{2n+1} \notin E$ and using that only strings in $\{0\}^*$ have to be queried by the Turing reduction, one obtains a set E that is hard for **EXP** with respect to positive truth-table reducibility.

Now let $\{y_0, y_1, \ldots\}$ be a set of strings such that y_n has length n and has maximal Kolmogorov complexity with respect to the time-bound 2^{2^n}. The set $A = \{a_n y_{|a_n|} : 0^n \in E\}$ is ptt-hard for **EXP** since $|a_n| \leq \log(n) + 2$ for all n and the equation

$$0^n \in E \Leftrightarrow (\exists y \in \{0,1\}^{|a_n|})\,[xy \in A]$$

needs at most $4n$ disjunctive queries.

Every string in A is of the form $xy_{|x|}$ for some x. As $y_{|x|}$ has at least Kolmogorov complexity $|x|$ with respect to the time bound $2^{2^{|x|}}$, it follows that the intersection of A and any given sparse set in **EXP** is finite. On the other hand, A is infinite and therefore **EXP**-avoiding and **EXP**-hyperimmune. □

Corollary 3.8 *There is a relativized world in which some* **EXP***-hyperimmune set is hard for* **EXP** *with respect to positive truth-table reducibility.*

It is still unknown whether an **EXP**-hyperimmune set can be hard for **EXP** with respect to the parity-reducibility in any relativized world. We show in the next section that the more restrictive notion of general generic sets does not permit **EXP**-hyperimmune sets to be hard for **EXP** with respect to parity-reducibility.

4 Immunity and Related Concepts

In this section, we investigate to which extent immunity notions are compatible with other well-known complexity theoretic properties such as randomness, genericity, approximability, and simplicity.

4.1 Randomness and Genericity

Within this section, the notions of **EXP**-Immunity are compared to the notions of resource-bounded randomness and genericity. These notions are effective variants of concepts from measure theory and Baire category. These notions need to be more restrictive than their classical counterparts in order to be meaningful on complexity classes as these classes are countable (except in the case of non-uniform classes like **P/poly**). In a classical sense countable classes are always small; that is, they have measure 0, and are meager. Lutz [16] introduced the following notion of measure 0 and random sets with respect to quasi-polynomial time computations. Quasi-polynomial means time $n^{\log^c(n)}$ for some constant c. This class corresponds to EXP if one uses inputs of exponential size (as is done in the case of functionals).

Definition 4.1 (Lutz [16]) *Let a_x be the x-th string with respect to the length-lexicographically ordered list $\lambda, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$ of all strings. That is, $a_0 = \lambda$, $a_1 = 0$, $a_2 = 1$, $a_3 = 00$, and so on. Furthermore, call f a* **Q**-functional *iff the domain of f is the set of prefixes of characteristic functions, and, for arbitrary sets A, the value $f(A(a_0)A(a_1)\ldots A_{a_x})$ is computed in quasi-polynomial time, that is, in time $x^{\log^c(x)}$ for some constant c.*

A **Q**-*functional f is a* **Q**-martingale *iff the values of f are (codes for) positive rational numbers and f satisfies,*

$$f(B(a_0)B(a_1)\ldots B(a_x)) \geq \frac{1}{2} \cdot (f(B(a_0)B(a_1)\ldots B(a_x)0) + f(B(a_0)B(a_1)\ldots B(a_x)1)),$$

for every set B, and every x. A **Q**-*martingale f* succeeds *on a set A iff, for every rational number r, there is an x such that $f(A(a_0)A(a_1)\ldots A(a_x)) > r$.*

A class has **Q**-measure 0 *iff there is a* **Q**-*martingale which succeeds on every set in the class. A set A is called* **Q**-random *iff no* **Q**-*martingale succeeds on the set A.*

Note that a class may fail to have **Q**-measure 0 although it does not contain **Q**-random sets. The most prominent example for such a class is the class **EXP** itself as on the one hand **EXP** does not have **Q**-measure 0 while on the other hand no set in **EXP** is **Q**-random [4, 15, 16]. The next proposition shows that **Q**-random sets cannot be **EXP**-hyperimmune. The proof uses a standard technique which can also be applied to show that general generic sets (as defined below) are not random [1].

Proposition 4.2 *No* **EXP**-*hyperimmune set is* **Q**-*random. In particular, the class of all* **EXP**-*hyperimmune sets has* **Q**-*measure* 0.

Proof. Let A be **EXP**-hyperimmune and let g be the function which maps any string of length n to the set $I_n = \{0^i1^j : i, j \geq 0 \wedge i + j = n\}$. The function g is honest and total. Thus there are infinitely many sets I_n disjoint to A. Now consider the following p-martingales f_n

defined by $f_n(\lambda) = 1$ and

$$f_n(b_0 b_1 \dots b_x b_{x+1}) = \begin{cases} 1.0 \cdot f_n(b_0 b_1 \dots b_x) & \text{if } x \notin I_n; \\ 1.5 \cdot f_n(b_0 b_1 \dots b_x) & \text{if } x \in I_n \text{ and } b_{x+1} = 0; \\ 0.5 \cdot f_n(b_0 b_1 \dots b_x) & \text{if } x \in I_n \text{ and } b_{x+1} = 1. \end{cases}$$

If $I_n \subseteq \overline{A}$ then f_n converges, on A, to the value 1.5^{n+1}. The sum over all 1.5^{-n-1} for $n = 0, 1, \dots$ is 2. Now let f be the p-functional given by the infinite sum $0.5 \cdot (1.5^{-1} \cdot f_0 + 1.5^{-2} \cdot f_1 + 1.5^{-3} \cdot f_2 + 1.5^{-4} \cdot f_3 + \dots)$, then each part $1.5^{-n-1} \cdot f_n$ is positive on every set and for those infinitely many n where A is disjoint to I_n the functional $1.5^{-n-1} \cdot f_n$ goes in the limit to 1. It follows that f diverges to ∞ on every **EXP**-hyperimmune A, and thereby witnesses that the class of all **EXP**-hyperimmune sets has **Q**-measure 0. □

Ambos-Spies, Fleischhack and Huwig [2] introduced a notion of genericity which is compatible with the notion of randomness in the sense that every random set is generic but not vice versa. Lutz [15] transferred the original general definition from Computability theory to complexity theory.

Definition 4.3 (Ambos-Spies, Fleischhack and Huwig [2]; Lutz [15])
A set A is **Q**-generic *iff, for every* **Q***-functional f,*

- *either $f(A(a_0)A(a_1)\dots A(a_x)) \notin \{0, 1\}$ for almost all x*
- *or $f(A(a_0)A(a_1)\dots A(a_x)) = A(a_{x+1})$ for infinitely many x.*

The condition $f(A(a_0)A(a_1)\dots A(a_x)) \notin \{0, 1\}$ permits f not to make a prediction. If f makes infinitely many predictions on A, then the second case must pertain.

A set A is general **Q**-generic*, if the functional is either almost always undefined, or it infinitely often predicts the next quasi-polynomially many values and one of these predictions is met by A. Predicting quasi-polynomially many values means that f predicts $A(a_{x+1})$ up to $A(a_{q(x)})$ where $q(x) = 2^{\log^c(x)}$ for some constant c.*

Ambos-Spies [1] showed that no general **Q**-generic set is **Q**-random. On the other hand, every **Q**-random set is still **Q**-generic [3, 4] so that these two notions of genericity are different. It follows from the definition that every **Q**-generic set is **EXP**-hyperimmune. As any **EXP**-avoiding set A contains only finitely many strings from the set $\{0\}^*$ one can easily show that A is not **Q**-generic by considering a function which predicts that every element of $\{0\}^* - A$ would be in A.

Fact 4.4 *Every general* **Q***-generic set is* **EXP***-hyperimmune. Some but not all* **Q***-generic sets are* **EXP***-hyperimmune. No* **EXP***-avoiding set is* **Q***-generic.*

The next Theorem on parity-hardness does not transfer to positive truth-table reducibility as we can build a general **Q**-generic set which is positive truth-table hard for **EXP**under the assumption that **EXP** $\subseteq$ **P/poly** (see Theorem 4.7).

Theorem 4.5 *No general* **Q***-generic set is hard for* **EXP** *with respect to parity-reducibility.*

Proof. Let E be the set in **EXP** constructed in Theorem 2.7 which is incompressible via parity-reducibility. Assume by way of contradiction that there is a parity reduction f from E to some **Q**-generic set A and let U be a polynomial time computable infinite set of indices of f. $F = \{e \in U$: there is some $x \in \{0,1\}^e$ such that $f\{x\}$ contains some string z with $|z| \geq |x|\}$. Let p be a polynomial such that $p(n)$ is an upper bound for the size $|y|$ of the largest $y \in f\{x\}$ with $x \in \{0,1\}^n$ for any given n.

Now one defines a **Q**-functional g which for any input of the form $A(\lambda)A(0)\ldots A(1^n)$ with $n+1 \in F$ predicts $A(y) = 0$ for all $y \in \{0,1\}^*$ with $n < |y| \leq p(n+1)$. This functional g is quasi-polynomial time computable and makes infinitely often predictions. Thus one of the predictions is satisfied by A.

It follows that there is an e such that $A(y) = 0$ for all $y \in \{0,1\}^*$ with $e \leq |y| \leq p(e)$. This implies that the characteristic function of $E \cap \{0,1\}^e$ can be computed using f and the information which z with $|z| < e$ are in A, contradicting the construction of E. Therefore no general **Q**-generic set can be hard for **EXP** with respect to parity-reducibility. □

Open Question 4.6 *Is there, in any relativized world, an* **EXP***-hyperimmune set that is hard for* **EXP** *with respect to parity-reducibility.*

Theorem 4.7 *In every relativized world where* **EXP** $\subseteq$ **P/poly** *there is a truth-table hard set for* **EXP** *which is general* **Q***-generic.*

Proof. Since **EXP** $\subseteq$ **P/poly** there is a set A such that $H = \{0^x : x \in A\}$ is **EXP**-complete. Consider the set $B = \{a_x 10^y : x \in A\}$. There is an enumeration $f_0, f_1, \ldots$ of **Q**-functionals such that every f_e on data of the form $L(a_0)L(a_1)\ldots L(a_x)$ with $n = |a_x| \geq 8$ only predicts L at places of length $n, n+1, \ldots, n^e$. We will now show how to modify B to obtain a set C which is general **Q**-generic. For the construction, let

$$U_n = \{n^{\log^k(n)} : k < \log\log(n)\}$$

for all $n \geq 8$, where $\log(m)$ is the smallest natural number o with $2^o \geq m$. Note that (a) for almost all n and all $k < \log\log(n)$ it holds that $n^{\log^k(n)} < 2^{n-10}$ and that (b) for all n the bound $e < \log(n)$ is satisfied. Moreover, one chooses the l_e to grow so fast that for every e and n with $n \geq 8$ and $l_e \leq \max(U_n)$ it holds that $2^{e+3} < |U_n|$.

Before stage 0**:** Let $B_0 = B$. Initialize all restraints r_e to l_e (they might be increased later).

Stage s**:** Let B_s be the current variant of B before entering stage s. Find the smallest $e \leq s$ such that f_e is a **Q**-functional satisfying the following two conditions:

- f_e on input from B_s does not make a correct prediction affecting only values below r_e,
- there are x and m such that $m = |a_x|$, $r_e + 8 \leq m \leq s$ and f_e with input $B_s(a_0)B_s(a_1)\ldots B_s(a_x)$ makes predictions on some set D_s (used below) of strings of length up to m^e.

There are two cases:

(I) If such an e is found, then let $B_{s+1}(a) = b$ if the function f_e predicted b at place a and $a \in D_s$ and let $B_{s+1}(a) = B_s(a)$ otherwise. Furthermore, the $r_{e'}$ with $e' \geq e$ are updated to the new value s^e.

(II) If no such e is found then let $B_{s+1} = B_s$ and do not change the restraints.

Let $C = \lim_{s\to\infty} B_s$. Whenever the set D_s exists in stage s and m is defined as above, then $|y| \leq m^e$ for all $y \in D_s$. The conditions $e < \log(n)$ and $m \leq \max(U_n)$ enable to show the implications

$$n^{\log^k(n)} \geq m \Rightarrow n^{e\cdot\log^k(n)} \geq m^e \Rightarrow n^{\log^{k+1}(n)} > m^e.$$

From these implications it follows that for every single prediction of $f_{e'}$ with $e' \leq e$ the set E of lengths of the places where predictions were made satisfies $|E \cap U_n| \leq 1$. It is easy to see that every requirement number e acts at most 2^e times and thus, for any x, at most 2^{e+1} strings of the form $a_x 10^{m-1-|a_x|}$ with $m \in U_n$ and $n = |a_x| + 8$ are changed. So $x \in A$ iff the majority of the strings $a_x 10^{m-1-|a_x|}$ with $m \in U_n$ and $n = |a_x| + 8$ is in C.

This implies that C is ptt-hard for **EXP** since the set $H = \{0^x : x \in A\}$ can be ptt-reduced to C. That this reduction is polynomial time computable follows from the fact that $|a_x| \leq \log(x) + 2$ and so the largest query has length $n^{\log^{\log\log(n)}(n)}$ with $n \leq \log(x) + 10$ which for almost all x is below x itself – note that x and not $\log(x) + O(1)$ is the length of the input 0^x.

It follows from the usual priority arguments, that the set C is **Q**-generic as every requirement e which can act infinitely often is eventually satisfied. Furthermore, including r_e into the list of the updated restraints $r_{e'}$ with $e' > e$ makes sure that the requirement e does not destruct its own work in later stages. □

Theorem 4.8 *There is a relativized world in which there is no Turing hard set for* **EXP** *which is general* **Q***-generic.*

Proof. Let $g(0) = 1$ and $g(n+1) = 2^{g(n)}$ for all n. Now the oracle C is constructed such that it only contains strings of the form $0^{g(n+1)-g(n)}x$ where n is a natural number and $x \in \{0,1\}^{g(n)}$. Furthermore, one chooses C at length $g(n+1)$ such that the finite set

$$A_n = \{x \in \{0,1\}^{g(n)} : 0^{g(n+1)-g(n)}x \in C\}$$

differs for all $e \leq n$ from all sets decided by the e-th Turing reduction with time bound $2^{g(n)} - 1$ relative to any oracle of the form $B \cup C$ where B does not contain strings longer than $\frac{1}{n}g(n)$. Note that due to the time constraint the oracle C is not queried at length $g(n+1)$ or beyond.

As there are less than $2^{g(n)/n+1}$ many strings in B of length up to $g(n)/n$ and as there are only n reductions considered, there are at most $n \cdot 2^{2^{g(n)/n+1}}$ many characteristic functions which A_n should not take. But as A_n can have 2^{2^n} many possible characteristic functions on the strings of length n, one can choose A_n as desired.

The union A of all A_n is in **EXP** relative to the oracle C by the choice of C: if x does not have length $g(n)$ for any n then $x \notin A$ and if x has length $g(n)$ then we can compute, in exponential time, the string $0^{g(n+1)-g(n)}x$ and ask whether this string is in the oracle C.

Assume now, by way of contradiction, that A is Turing reducible to a **Q**-generic set B via φ_e^B in time $p(n)$ where p is a polynomial. Consider the **Q**-functional which predicts for every set D on input $D(0)D(1)\ldots D(1^{g(n)/n-1})$ that all further values of D up to $D(1^{p(g(n))})$ are 0 if $e < n \wedge p(g(n)) < g(n+1)$ and which is undefined otherwise. There are infinitely many predictions and so the set B meets one of them, belonging to some $g(n)$. Now it follows that φ_e computes the values $A_n(x)$ for all $x \in \{0,1\}^{g(n)}$ from the entries of the oracle C of length $0, 1, \ldots, g(n)$ and from the entries of the set B which have at most length $g(n)/n$. All other entries queried are 0 and can be ignored. Furthermore, φ_e respects the time bound $2^{g(n)} - 1$. As $n > e$, A_n had been chosen previously such that this computation does not give A_n and this contradiction gives that either B is not Turing hard for **EXP** in the world relative to C or that B is not **Q**-generic. □

4.2 Approximability

We can ask how immunity notions for exponential time relate to the notions of approximability of sets as defined by Beigel, Kummer and Stephan [7] where a set is called *approximable* iff there is a **P**-function f and a constant k such that for every input $x_1, x_2, \ldots, x_k$ the function f computes in polynomial time k bits $y_1, y_2, \ldots, y_k$ such that one of these bits coincides with the characteristic function of A: $y_l = A(x_l)$ for some $l \in \{1, 2, \ldots, k\}$. We consider two special cases of approximability: a set $\{b_0, b_1, \ldots\}$ is **P**-*retraceable* iff there is a **P**-function f with $f(a_x) = a_y$ for some $y \leq x$ and all natural numbers x and $f(b_{n+1}) = b_n$ for all n. A **P**-*semirecursive set* is a set B where one can compute from any finite set D of strings in time polynomial in the sum of their lengths an input string a member $x \in D$ such that either $x \in B$ or $D \subseteq \overline{B}$. Note that **P**-retraceable and **P**-semirecursive sets are both approximable with the constant k having the value 2.

Theorem 4.9 *An* **EXP***-avoiding set can be* **P***-retraceable but it is never* **P***-semirecursive.*

Proof. To see that an **EXP**-avoiding set can be **P**-retraceable, consider the retracing-function that maps every string xy satisfying $2|x| - 1 \leq |xy| \leq 2|x|$ to x on a set A which contains the strings $x_0 x_1 x_2 \ldots x_n$ where $x_0 = 0$ and x_{n+1} is defined inductively as a string in $\{0,1\}^m$ for which the conditional Kolmogorov complexity $K(x_{n+1} | x_0 x_1 x_2 \ldots x_n)$ is maximal with respect to computations needing time 2^{2^m}. As the elements of sparse sets in **EXP** always have small double exponential Kolmogorov complexity, every sparse set in **EXP** contains only finitely many elements of A.

Let B be any **P**-time semirecursive set. Then one can compute for any length n in exponential time a string x_n such that $x_n \in B$ whenever B has any elements of length n. It follows directly that B has an infinite intersection with the sparse set $E = \{x_0, x_1, x_2, \ldots\}$ whenever B is infinite. □

Note that the second result also holds with "**EXP**-semirecursive" in place of "**P**-semirecursive".

The second result cannot be strengthened to **EXP**-hyperimmune sets. Dekker constructed a set A which is hypersimple and semirecursive in the computability theoretic sense [17, Theorem II.6.16 and Theorem III.3.13]. An easy modification of the construction makes the set A **P**-semirecursive. The complement $\overline{A}$ of A is then also **P**-semirecursive. Furthermore, $\overline{A}$ is **EXP**-hyperimmune as $\overline{A}$ is already hyperimmune in the sense of computability theory.

4.3 Simplicity

NEXP-simple sets are infinite sets in **NEXP** with **NEXP**-immune complement. Such sets can be defined in a very general way.

Definition 4.10 ([6]) *Let $\mathcal{C}$ be a class of languages. A set A is called $\mathcal{C}$-immune, if it is infinite and does not contain any infinite subset in $\mathcal{C}$. A set A is called $\mathcal{C}$-simple, if A is infinite, $A \in \mathcal{C}$ and the complement $\overline{A}$ is $\mathcal{C}$-immune.*

Simplicity has been studied at many levels, ranging from computability [11, 17, 18, 19] to complexity theory [5, 14, 20, 21]. A major open problem is the existence of simple sets under some natural assumptions (maybe involving measure theory). Easy padding arguments give us the following relations between complexity-theoretic variants of simple sets. If there is a **NEXP**-simple set, then there is a **NE**-simple set (namely $\{x : x[1 \ldots |x|^{1/k}] \in L\}$ where $L \in \mathbf{NTIME}(2^{n^k})$ is **NEXP**-simple). If there is an **NE**-simple set, then there is an **NP**-simple set in $\mathbf{P}/1$ ($\{x : |x| = y, y \in L\}$, where L is **NE**-simple).

For any complexity class $\mathcal{C}$ that allows us to run Ladner's delayed diagonalization technique we can show that any $\mathcal{C}$-simple degree bounds a $\mathcal{C}$-simple degree which is Turing-incomplete for $\mathcal{C}$. For example, any **NP**-simple degree bounds a degree Turing-incomplete for **NP**, which contains an **NP**-simple set (the same is true for **NEXP**). To prove this, we use Ladner's construction to construct a set $A \in P$ such that $S \cup A$ is Turing-incomplete for $\mathcal{C}$, and remains $\mathcal{C}$-simple. Of course $S \cup A \leq_{\mathrm{T}} S$. It might be worth to try to exploit these ideas in order to attack the following open problem.

Open Question 4.11 *Is there a relativized world in which* **NEXP***-simple sets exist but none of them is Turing hard for* **NEXP***.*

5 Conclusion

The central topic of the paper is the question, for which reducibilities can **EXP**-immune, **EXP**-hyperimmune and **EXP**-avoiding sets be hard for **EXP**. With respect to many-one and conjunctive reducibilities, Theorem 3.1 shows that no **EXP**-immune set can be hard for **EXP**, which implies the same result for the more restrictive notions of **EXP**-hyperimmune and **EXP**-avoiding sets. With respect to disjunctive reducibility, Buhrman [9] constructed an **EXP**-immune hard set for **EXP**, while Theorem 3.3 states that there are no **EXP**-hyperimmune and thus also no **EXP**-avoiding sets which are hard for **EXP**. These results were obtained by using Theorem 2.7 which states that **EXP**-complete sets cannot be compressed conjunctively or disjunctively. Although Theorem 2.7 also covers the case of parity-reducibility (= linear reducibility), it is still unknown whether an **EXP**-hyperimmune set can be hard for **EXP** with respect to parity-reducibility.

Open Question 5.1 *Is there a relativized world in which there is an* **EXP***-hyperimmune set that is hard for* **EXP** *with respect to parity-reducibility?*

In the case of positive truth-table reducibility one has, on the one hand, a relativized world with an **EXP**-hyperimmune, even **EXP**-avoiding set while, on the other hand, Theorem 3.4

states that, for any $\alpha < 1$ and in any relativized world there is no **EXP**-hyperimmune set with respect to truth-table reductions asking n^α many queries. Theorem 3.6 states that there is an **EXP**-avoiding Turing-hard set for **EXP** with respect to reductions using $\log^2(n)$ queries while as an immediate consequence of Theorem 3.4 we know that no **EXP**-hyperimmune Turing-hard set for **EXP** with respect to reductions using $\alpha \log(n)$ queries where $\alpha < 1$. Finally, Theorem 4.7 shows that general **Q**-generic sets can be positive truth-table hard for **EXP** in those relativized worlds where $\mathbf{EXP} \subseteq \mathbf{P/poly}$ while there are, by Theorem 4.8, other relativized worlds where **Q**-generic sets are even not Turing-hard for **EXP**.

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