Firing Synchronization on the Ring

André Berthiaume, Ljubomir Perkovic, Amber Settle *
DePaul University

Janos Simon†
University of Chicago

January 17, 2002

Abstract

We present two solutions to the firing synchronization problem on the ring, an 8-state minimal-time solution and a 6-state non-minimal-time solution. Both solutions use fewer states than the previous best-known minimal-time automaton, a 16-state solution due to Culik. We also give the first lower bounds on the number of states needed for solutions to the ring firing synchronization problem. We show that there is no 3-state solution and no 4-state, symmetric, minimal-time solution for the ring.

1 Introduction

In the original firing synchronization problem we consider a one-dimensional array of n identical finite automata. Initially all automata are in the same state except for one automaton that is designated as the initiator for the synchronization. The machines operate in lock-step, and the transitions of each automaton depend on the state of the automaton and the states of its neighbors. The goal is to define the set of states and transition rules for the automata so that all machines enter a special fire state for the first time and simultaneously during the final round of the computation. A great deal of work has been done on the original firing synchronization problem [1, 4, 6, 7, 8, 12, 14, 15, 18, 19].

*DePaul University, School of Computer Science, Telecommunications and Information Systems, 243 S. Wabash, Chicago, IL, 60604, FAX: (312) 362-6116, E-mail: {aberthia, lperkovic, asettle}@cti.depaul.edu

†University of Chicago, Department of Computer Science, 1100 E. 58th Street, Chicago, IL 60637, FAX: (773) 702-8487, E-mail: simon@cs.uchicago.edu

There are many variations of the firing synchronization problem that involve networks of automata other than the one-dimensional array [3, 5, 9, 10, 11, 15, 16, 17]. We consider the problem of synchronizing rings of finite automata. In this problem each automaton has exactly two neighbors and there are no endpoints in the system. The goal is the same as the original problem, namely the synchronization of all automata in the final round of the computation.

Initial work on the ring variant of the firing synchronization problem focused on finding correct solutions to the problem without considering the number of states or even the minimal-time required to solve the problem [10, 5, 3, 11]. The main reason for this is that the solutions to the ring were given as an initial step in solving a more general problem, that of synchronizing connected graphs. The solution to the ring was not the goal, but a necessary first step. Further details about these solutions can be found in the next section.

The first work directly considering the number of states needed to solve ring synchronization was done by Culik [2]. He established the minimal-time necessary for synchronization and produced a minimal-time solution using 16 states. He did so in the context of solving a related synchronization problem in which there are multiple initiators.

In this paper we improve Culik's result for the firing synchronization problem on a ring by giving an 8-state, symmetric, minimal-time solution, as well as a 6-state, non-minimal-time solution for the ring. We also give the first state lower bounds known for the firing squad problem on the ring. We show that there is no 3-state solution, regardless of the time provided for synchronization. This result is the first known lower bound with no assumption about the time needed for synchronization. We also show that no 4-state, symmetric, minimal-time solution to the ring exists.

2 Preliminaries

We now outline the definitions for the ring version of the firing synchronization problem, sketch the previous work done on the problem, and state our results.

2.1 Definitions

One of the oldest variants of the firing synchronization problem is one in which the underlying network is not a one-dimensional array but a ring. As in the original firing problem there is a single initiator that may be located anywhere in the ring. The automata change state once during each round based on their current state and their neighbors' current state. The problem is to define the set of states and the transition function for the automaton so that all machines fire for the first time and simultaneously in some round $t(n)$.

The transition function for each automaton can be given as a set of 4-tuples. The 4-tuple (X,Y,Z,W) represents the rule that an automaton currently in state Y, with left neighbor in state X and right neighbor in state Z will enter state W at the next time step. We will denote this by XYZ $\rightarrow$ W. By definition automata solving the firing synchronization problem are deterministic so that there is at most one 4-tuple (X,Y,Z,W) for any triple of states X,Y,Z.

A *symmetric* automaton is one which has a symmetric transition function, that is, whenever a transition XYZ $\rightarrow$ W is defined, the transition ZYX $\rightarrow$ W must also be defined. This means that the automata cannot distinguish their left and right neighbors.

2.2 Previous work

Recall that initial work on the synchronization of the ring was done while developing solutions for connected graphs. Each of the papers that follows provides a solution to the ring as a building block in a more general solution. The first result of this type was given by Nishitani and Honda [10] in 1981. Their solution required $3r_G + 1$ steps to synchronize a connected, undirected graph where r_G is the maximum distance between the initiator and any other node in the graph. Kobayashi [5] adapted Nishitani and Honda's solution to directed graphs, giving a solution requiring $(2a^2 + 1)^n$ time where a is the number of output terminals of the automaton and n is the number of nodes in the graph. The first polynomial-time solution for directed graphs was given by Even, Litman, and Winkler [3] in 1990. They found a solution requiring $O(n^2)$ time to synchronize a strongly-connected directed graph with n nodes. This was further improved by Ostrovsky and Wilkerson [11], who gave a solution that synchronized a strongly-connected directed graph in time $O(nD)$ where n is the number of nodes in the graph and D is the diameter of the graph.

The first work directly considering the number of states needed to solve ring synchronization was done by Culik [2]. In his paper Culik considered a variation of the firing synchronization problem in which there are multiple initiators. He showed that any solution to the problem for the one-dimensional array of length n with two initiators located at the endpoints requires $n - 1$ steps to synchronize. The following theorem is a direct corollary of this result.

Theorem 2.1 (Culik) *Any solution to the firing synchronization problem for the ring with n automata requires n time steps to synchronize.*

If we take an array of length $n + 1$ with two initiators at the endpoints and consider it as a ring with a single initiator we obtain Theorem 2.1. It should be noted that Culik incorrectly gave a time bound of $n-1$ time steps in his paper, as he neglected to account for the fact that the length of the ring is shorter by one than the equivalent array since the two initiators are merged into one.

We will call any solution that synchronizes a ring of n automata in $t(n) = n$ time steps a *minimal-time solution.* Any solution that requires $t(n) > n$ time steps to synchronize will be called a *non-minimal-time solution.*

In addition to giving a time bound, Culik also described a minimal-time solution to the ring version of the problem. He used a modified version of Waksman's solution [18], producing an automaton that uses 16 states.

No prior work has been done on improving the number of states used by the minimal-time solution described above. Further, no state lower bounds for the ring version of the problem are known. In particular, the known state lower bounds for the original firing synchronization problem [1, 12] do not apply to the ring, and no one has directly considered state lower bounds for the ring.

2.3 Our contributions

We present a 8-state, symmetric, minimal-time solution to the firing synchronization problem on the ring. This solution is adapted from Szwerinski's solution to the original firing synchronization problem [16]. To synchronize an array of length n the ring solution requires time n.

We also give a 6-state, non-minimal-time solution for ring synchronization. This solution is an extension of Mazoyer's solution to the original problem [6]. It requires $2n - 2$ steps to synchronize an array with n automata.

We also give the first known state lower bounds for the firing synchronization problem on the ring. Our first state lower bound result is to prove the following theorem:

Theorem 2.2 *There is no* 3*-state solution to the firing synchronization problem for the ring.*

As noted in the previous section, this is the first known lower bound for the firing synchronization problem that places no restrictions on the time required to synchronize. With two additional conditions, we can extend the theorem to the following:

Theorem 2.3 *There is no 4-state, symmetric, minimal-time solution to the firing synchronization problem for the ring.*

Since the ring provides an inherently symmetric setting, it is natural to consider symmetric solutions for the ring.

3 The minimal-time solution

Our 8-state, minimal-time solution is adapted from Szwerinski's 8-state, symmetric solution to the firing synchronization problem on the one-dimensional array [16]. The construction of the solution requires the addition of some transitions to the solution, as well as the removal of transitions that are not needed for the ring, but the solution behaves in the same manner as Szwerinski's.

In Szwerinski's solution, the array is repeatedly subdivided into halves as new initiators are placed in the center(s) of each of the intervals. The simulation ends when all automata become initiators and then fire. The synchronization begins when the first initiator sends out a signal, the purpose of which is to produce a second initiator when it reaches the opposite end of the array. When this wake-up signal is reflected back by the new initiator it intersects with markers created in the wake of the first signal and produces a third initiator (or pair of initiators depending on the parity of the original array) located at the center of the array. This division of the array into halves continues until every other automaton is an initiator. At the next step in the simulation every automaton becomes an initiator, and at the next time step all automata fire. For a more detailed description of Szwerinski's solution, see the references [13, 16].

Because the eight-state solution is symmetric, it can be adapted to the ring in a straightforward manner. Instead of a single wake-up signal, two signals are sent from the initiator. These intersect on the opposite side of the ring, creating either one or two new initiators depending on the parity of the ring. The process then continues as described above until every automaton becomes an initiator and then fires.

To produce the above results on the ring, two types of changes to the transition function had to be made. First, all unnecessary transitions were eliminated. A transition is unnecessary if it involves a triple that does not appear in any simulation. Clearly, any transition involving the end marker is unnecessary, as the end marker is used in solutions to the original problem to indicate the end of the array. The marker allows the definition of a single transition function instead of three different types of transition functions, one for the central automata and one for each of the left and right end machines. Since there are no endpoints in the ring, these transitions can be removed. In addition, the transitions ARA $\rightarrow$ Q, PRQ $\rightarrow$ Q, QRP $\rightarrow$ Q, and QRQ $\rightarrow$ Q were eliminated. Each corresponds to a configuration produced only for arrays.

Next, additional transitions had to be defined for configurations that appeared in simulations on the ring but did not appear in any simulations in the array. These transitions are AZA $\rightarrow$ G, AAR $\rightarrow$ G, RAA $\rightarrow$ G, AAP $\rightarrow$ G, PAA $\rightarrow$ G, AAG $\rightarrow$ G, GAA $\rightarrow$ G, QGG $\rightarrow$ G, and GGQ $\rightarrow$ G. All of these configurations represent triples produced late in the simulation and are of two types. The first is triples produced immediately following the creation of the center initiator or initiators, and the second is triples that occur just prior to synchronization. The state A is used in Szwerinski's solution as a pseudo-initiator to break symmetry in these places in the simulation. Because the ring produces more symmetric behavior than the array, more transitions designed for this purpose were needed.

Table 1 shows the transition function for the 8-state automaton. The state of an

neighbors' states	Z	A	B	R	P	Q	G
	present state						
Z–Z	Z	Z	B				G
Z–A	A	Z	G	A			
Z–B	Z	G	B		P		
Z–R	R	P	P	Q	R	Z	G
Z–P	Z	R	B	Q			
Z–Q	Z		B	Q	R		
Z–G	A	R		B	A		G
A–A	G	G					G
A–B	A	G	G	G	P		
A–R	P	G				A	
A–P	R	G	G	Q			
A–Q	A		G		P		
A–G	R	G	G	B			G
B–B	Z				P		G

neighbors' states	Z	A	B	R	P	Q	G
	present state						
B–R	R	P	P		R	Z	G
B–P	Z	R				Q	
B–Q	Z			B	R		
B–G	A	R	B		A		G
R–R		P					G
R–P		R	Q			Z	
R–Q		P				Z	G
R–G		R	B	A		A	G
P–P							A
P–Q	Z	R				Z	
P–G			B	A		A	A
Q–G	A	R			A		G
G–G	G		G	G			F

Table 1: The transition function for the 8-state automaton

automaton at the next time step can be found by looking at the entry in the column corresponding to the automaton's present state and the row corresponding to the states of its neighbors. Since the automaton is symmetric, the orientation of the neighbors is irrelevant.

4 A non-minimal-time solution

The 6-state, non-minimal-time ring solution is an extension of Mazoyer's 6-state solution to the restricted firing synchronization problem [6]. The restricted version of the problem requires that the initiator to be located at the left endpoint of the array. The construction of a ring solution requires a slight modification of the transition function, but the solution behaves in the same manner as Mazoyer's.

Mazoyer's solution works by dividing the line of n automata into unequal parts, one of length $\frac{2}{3}n$ and the other of length $\frac{1}{3}n$. An initiator is placed at the left end of the shorter segment, and each segment is then recursively subdivided. After every automaton becomes an initiator, the automata fire and the synchronization ends. For a detailed description of the solution see Mazoyer's paper [6].

In order to extend Mazoyer's solution to the ring, two types of changes had to be made. First, all transitions involving the end marker were eliminated, as in with the 8-state solution described above. Next, two transitions were added. These transitions are ZGZ $\rightarrow$ A and ZZA $\rightarrow$ Z. The transition ZGZ $\rightarrow$ A is needed to prevent a wake-up signal from propagating to the left of the first initiator. At the very next time step, the transition ZZA $\rightarrow$ Z must be defined in order to keep all the initiators to the left of

A	Z	A	B	C	G
Z		A	Z	G	
A	A	A	B	C	B
B	G		G	C	C
C	A	A			
G				C	C

B	Z	A	B	C	G
Z		G	B	Z	B
A	G	B	B	Z	
B	G	A	B	C	B
C	Z	A			Z
G	C	C		B	G

C	Z	A	B	C	G
Z	C	A	G	C	G
A	B		B		B
B	C			C	G
C	C	A	B	C	B
G	B		B		B

Z	Z	A	B	C	G
Z	Z	Z	Z	Z	Z
A	G	Z	Z	Z	C
B	Z	Z	Z	Z	Z
C	A	Z	Z	Z	G
G	C	Z	Z	Z	A

G	Z	A	B	C	G
Z	A	G	G	G	
A	B		G	G	
B	B		G	G	G
C	A		G	G	A
G	B		G	G	F

Table 2: The transition function for the 6-state automaton

the first initiator quiescent. The purpose of both of these transitions is to preserve the behavior of Mazoyer's solution.

The fact that a solution to the restricted firing synchronization problem could be adapted to work on a ring is remarkable. Particularly interesting in this case is that Mazoyer's solution is distinctly non-symmetric. He relied on asymmetry to help him reduce the number of states needed for the solution, which is why the solution only works for the restricted version of the original problem. Despite this, the solution could be modified to work on the ring, where symmetry is inherent. We conjecture that this is a consequence of the structure of Mazoyer's solution, and that not all non-symmetric solutions can be modified for the ring.

Table 2 gives the transition function for the 6-state non-minimal-time automata. The state of an automaton at the next time step can be found by looking at the table corresponding to the automaton's present state. The state that the automaton should enter at the next time step is the one in the row and column corresponding to the states of its left and right neighbors respectively.

5 Lower bounds

As mentioned previously, there are no known lower bounds for the firing synchronization problem on a ring. In this section we show that there is no 3-state solution to ring synchronization. We also show that there is no 4-state, symmetric, minimal-time solution.

5.1 Three-state bound

We now prove Theorem 2.2, a result stating that there is no 3-state solution to the firing synchronization problem on the ring.
Proof: Denote the three states by G, Z, and F. Since there are only three states for the solution and the fire state cannot be used prior to the final round, there are only eight possible triples of states that may be used prior to the last round. These are: ZZZ, GZZ, ZGZ, ZZG, GZG, GGZ, ZGG, and GGG. We know that ZZZ $\rightarrow$ Z must be defined. Partition the triples into four classes, based on the number of initiators.

Class 0	ZZZ
Class 1	GZZ, ZGZ, ZZG
Class 2	GGZ, GZG, ZGG
Class 3	GGG

Consider the ring of length 3. By assumption, the initial configuration is ZGZ. In order to produce the next configuration we must apply three class 1 rules. The next configuration, however, must have at least two initiators, since otherwise it would duplicate the initial configuration.

This means that there are two cases to consider:

1. Class 1 rules have all G's on the right hand side, or

2. Exactly two of the class 1 rules have a G on the right hand side.

In the first case, we must have GGG $\rightarrow$ F. This yields a contradiction for the ring of length four, where after one round we produce the configuration GGGZ.

In the second case, we must have all class 2 triples defined to have G on the right hand side, or we produce an infinite loop for the length three ring. This is because in the length three ring, the first configuration has two Z's and the second configuration has one Z. Since it is not possible by the definition of the problem to have three Z's, the next configuration must have no Z's.

So in the case where exactly two of the class 1 triples are defined to transition to G, all class 2 rules must use G on the right hand side. Further, we must have GGG $\rightarrow$ F must be defined, since it is the only remaining undefined triple. Consider the ring of length five. ZZZGZ yields ZZs, where s is a string with 2 initiators. If the initiators are adjacent in s, the we are done since in the next round we get the triple GGG from the triples ZGG, GGZ, and either ZZG or GZZ since two out of three of the class 1 triples are defined to transition to G. This produces a firing prior to the final round. If the string s is of the form GZG, so that the length 5 ring in round 2 looks like ZZGZG, we must have had ZGZ $\rightarrow$ Z, ZZG $\rightarrow$ G, and GZZ $\rightarrow$ G. This produces the configuration GGZGZ and then GGGZG, which causes a partial firing of the ring contrary to the definition of the problem. $\diamondsuit$

5.2 Four state bound

Recall that Theorem 2.3 states that there is no 4-state, symmetric, minimal-time symmetric solution to the ring synchronization problem. We now give the proof of the theorem.

Proof of Theorem 2.3

This result follows from the fact, shown by Balzer [1] and verified by Sanders [12], that there is no 4-state minimal-time solution to the firing synchronization problem on an array, and the following lemma:

Lemma 5.1 *If there exists a symmetric, minimal-time k-state solution to the firing squad problem on a ring, then there exists a symmetric, minimal-time k-state solution to the firing squad problem on an array.*

To see intuitively why this lemma is true, we describe how to construct a simulation on an array of $n = 6$ automata from a simulation on a ring of $2n - 2 = 10$ automata. We first run a simulation on the ring, using the symmetric, minimal-time k-state solution to the firing synchronization problem on a ring:

0	G	Z	Z	Z	Z	Z	Z	Z	Z	Z
1	?	?	Z	Z	Z	Z	Z	Z	Z	?
2	?	?	?	Z	Z	Z	Z	Z	?	?
3	?	?	?	?	Z	Z	Z	?	?	?
4	?	?	?	?	?	Z	?	?	?	?
5	?	?	?	?	?	?	?	?	?	?
6	?	?	?	?	?	?	?	?	?	?
7	?	?	?	?	?	?	?	?	?	?
8	?	?	?	?	?	?	?	?	?	?
9	?	?	?	?	?	?	?	?	?	?
10	F	F	F	F	F	F	F	F	F	F

We obtain the simulation on an array of $n = 6$ automata by simply removing the last four columns.

In the arguments below, we will assume that $2n-2$ automata on a ring are numbered 1 through $2n-2$ in counter-clockwise order, with the initiator being numbered 1. Before we formally prove Lemma 5.1, we first show the following holds:

Claim 5.1 *Suppose we run a simulation of a symmetric solution on a ring of $2n-2$ automata. Then, in any round r, automata i and $2n-i$ must be in the same state, for $i = 2, 3, ..., n-1$.*

Proof: We use induction on r. If $r = 0$, the claim holds trivially since all relevant automata are quiescent. Consider now round $r \geq 1$ and choose some i between 2 and $n-1$. By induction, automata $i-1$ and $2n-i+1$ (or 1 if $i = 2$) are in the same state in round $r-1$, as are automata i and $2n-i$, and automata $i+1$ and $2n-i-1$. Since the solution is symmetric, this implies that automata i and $2n-i$ must be in the same state in round r. This is true for any i between 2 and $n-1$, which completes the induction step. $\diamondsuit$

We are now ready to complete the proof of Lemma 5.1.

Proof: From our intuitive example, it should be clear that all we need to do is define the additional transitions for the array solution that involve the left or right end markers. Let δ_l be the set of all the transitions of the k-state, symmetric, minimal-time ring solution that are used by automaton 1 in a ring of size $2n-2$, for any $n \geq 2$. Each transition in δ_l must be of the form $XYX \rightarrow W$, since automata 2 and $2n-2$ are always in the same state, by the above claim. For each such transition, we define a new, array transition $*YX \rightarrow W$. Next, we consider the set δ_r of transitions of the k-state, symmetric, minimal-time ring solution that are used by automaton n in a ring of size $2n-2$, for any $n \geq 2$. Each transition in δ_r is also of the form $XYX \rightarrow W$. For each such transition, we define a new array transition $XY* \rightarrow W$. $\diamondsuit$

It should be noted that the symmetric solution requirement in Theorem 2.3 and Lemma 5.1 is stronger than necessary. In the proof of Lemma 5.1, all we really used is that the following two conditions are satisfied by the k-state, minimal-time solution for the firing synchronization problem on a ring:

1. For any simulation on a ring of even length $2n-2$, automaton 1 does not use two transitions $X_1YZ \rightarrow W_1$ and $X_2YZ \rightarrow W_2$ where $X_1 \neq X_2$ and $W_1 \neq W_2$.

2. For any simulation on a ring of even length $2n-2$, automaton n does not use two transitions $XYZ_1 \rightarrow W_1$ and $XYZ_2 \rightarrow W_2$ where $Z_1 \neq Z_2$ and $W_1 \neq W_2$.

6 Conclusion

In this paper we presented improved bounds on the complexity of solutions to the firing synchronization problem on the ring. We gave an 8-state, symmetric, minimal-time solution, as well as a 6-state, non-minimal-time solution to the firing synchronization problem on the ring. Both of these solutions use fewer states that the best-known solutions to the firing synchronization problem on the ring, a 16-state solution given by Culik [2].

We also provide the first lower bounds for the synchronization of the ring. We show that there is no 3-state solution to ring synchronization, the first known result to place no restrictions on the time needed to synchronize. We also prove that there is no 4-state, symmetric, minimal-time solution.

This work establishes a gap between the best-known upper bounds and lower bounds for ring synchronization. For minimal-time solutions this gap is 4 states in the symmetric case and 5 states in general. For non-minimal-time solutions the gap is only 3 states. Reducing this gap, either by producing a smaller solution for the ring or by improving the lower bounds, is an important direction for future work.

References

[1] R. Balzer. An 8-state minimal time solution to the firing squad synchronization problem. *Information and Control*, 10:22–42, 1967.

[2] K. Culik. Variations of the Firing Squad Problem and Applications. *Information Processing Letters*, 30:153–157, 1989.

[3] S. Even, A. Litman, and P. Winkler. Computing with snakes in directed networks of automata. *Journal of Algorithms*, 24(1):158–170, 1997.

[4] E. Goto. A minimum time solution of the firing squad problem. Course Notes for Applied Mathematics 298, Harvard University, 1962.

[5] K. Kobayashi. The firing squad synchronization problem for a class of polyautomata networks. *Journal of Computer and System Sciences*, 17(3):300–318, 1978.

[6] J. Mazoyer. A six-state minimal time solution to the firing synchronization problem. *Theoretical Computer Science*, 50:183–238, 1987.

[7] M. Minsky. Computation*: Finite and Infinite Machines*. Prentice Hall, 1972.

[8] E. F. Moore. The firing squad synchronization problem. In *Sequential Machines - Selected Papers*, pages 213–214. Addison-Wesley, 1964.

[9] F. R. Moore and G. G. Langdon. A generalized firing squad problem. *Information and Control*, 12:212–220, 1968.

[10] Y. Nishitani and N. Honda. The firing squad synchronization problem for graphs. *Theoretical Computer Science*, 14(1):39–61, 1981.

[11] R. Ostrovsky and D. Wilkerson. Faster computation on directed networks of automata. In *Proceedings of 14th Annual ACM Symposium on Principles of Distributed Computing*, pages 38–46, 1995.

[12] P. Sanders. Massively parallel search for transition-tables of polyautomata. In C. Jesshope, V. Jossifov, and W. Wilhelmi, editors, *Proceedings of the VI International Workshop on Parallel Processing by Cellular Automata and Arrays*, pages 99–108. Akademie, 1994.

[13] A. Settle. A symmetric nine-state automaton for the generalized firing synchronization problem. Technical Report 97–03, University of Chicago Department of Computer Science, 1997.

[14] A. Settle and J. Simon. Improved bounds for the distributed firing synchronization problem. In SIROCCO 5: Proceedings of the 5th International Colloquium on Structural Information and Communication Complexity, Amalfi, Italy, pp. 66-81, Carleton Scientific, 1998.

[15] T. A. Settle. New bounds for the distributed firing synchronization problem. Ph.D. thesis, Department of Computer Science, University of Chicago, 1999.

[16] H. Szwerinski. Time-optimal solution of the firing-squad-synchronization-problem for n-dimensional rectangles with the general at an arbitrary position. *Theoretical Computer Science*, 19:305–320, 1982.

[17] V. I. Varshavsky, V. B. Marakhovsky, and V. A. Peschansky. Synchronization of interacting automata. *Mathematical Systems Theory*, 4:212–230, 1970.

[18] A. Waksman. An optimum solution to the firing squad synchronization problem. *Information and Control*, 9:66–78, 1966.

[19] J. B. Yunès. Seven-state solutions to the firing squad synchronization problem. *Theoretical Computer Science*, 127:313–332, 1994.