Decidability of String Graphs

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Abstract

We show that string graphs can be recognized in nondeterminsitic exponential time by giving an exponential upper bound on the number of intersections for a drawing realizing the string graph in the plane. This upper bound confirms a conjecture by Kratochvíl and Matoušek [KM91] and settles the long-standing open problem of the decidability of string graph recognition (Sinden [Sin66], Graham [Gra76]).

1 Introduction

Is it possible that some A is B, some B is C, but no A is C? Easily, you say, and your mind conjures up a diagram that Euler (and Leibniz, and Sturm before him) used to illustrate this situation.

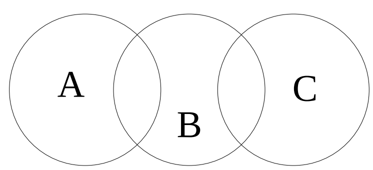

Figure 1: Some A is B, some B is C, but no A is C.

However, it is not always possible to illustrate a situation that is logically possible in an Euler diagram if we restrict ourselves to path-connected[1] regions in the plane: we can turn the complete graph on five vertices, into such an example with fifteen regions (one for each vertex and edge) [GPP95]. How can we determine whether an Euler diagrams exists for a given set of specifications?

Diagrammatic reasoning is concerned with representability of logical relations in the plane (or other spaces). This area has drawn attention from different research groups including Artificial Intelligence and Geometrical Information Systems [All83, GPP95], Spatial Databases [PSV99], Integrated Circuits [Sin66], and Logic [GPP95, LP97]. One of the major open problems in this area is the decidability of the existence of a representation for a given (satisfiable) formula. In particular the special case we considered above in which we specified for a collection of regions whether they should intersect or not has been open since the sixties.

[1] The situation changes if we use connected instead of path-connected. See Kratochvíl [Kra91a, Section 2].

This special case is captured by the combinatorial notion of string graphs. String graphs are intersection graphs of curves in the plane with a vertex for each curve and an edge representing an intersection between two curves. The notion was introduced in 1966 by Sinden [Sin66] who stated the main problem thus:

> It is specified which pairs of a collection of curves (or connected regions) in the plane cross and which pairs do not cross. When are such specifications consistent?

Sinden was working on the layout problem of integrated circuits (thin film RC circuits to be precise), and the string graph problem arose naturally in this context, since the technology for creating the circuits made it possible for some pairs of conductors to cross. On the theoretical side he observed that all planar graphs are string graphs, and also gave a small example of a graph which is not a string graph. Ron Graham, in 1976, introduced the problem to the combinatorial community [Gra76], and in the same year Ehrlich, Even, and Tarjan [EET76] showed that computing the chromatic number of string graphs is **NP**-hard. Since then string graphs have become a popular subject in graph theory, and several characterizations of special kinds of string graphs are known, but the general recognition problem remained open. In 1991 Kratochvíl [Kra91b] proved that the problem of recognizing string graphs is **NP**-hard showing that a characterization is not going to be polynomial time computable (unless $\mathbf{P} = \mathbf{NP}$). At the same time Kratochvíl and Matoušek [KM91] proved the surprising result that some string graphs require an exponential number of intersections to be realized in the plane. They conjectured an exponential upper bound on the number of intersections. We show that this conjecture is indeed true, putting the recognition problem of string graphs in **NEXP**.

2 Preliminaries

Given a graph $G = (V, E)$ and a set $R \subseteq \binom{E}{2} = \{\{e, f\} : e, f \in E\}$ on E, we call a drawing D of G in the plane a *weak realization* of (G, R) if only pairs of edges which are in R are allowed to intersect in D (they do not have to intersect, however). In this case we call (G, R) *weakly realizable*. We say that D is a *realization* of G if exactly the pairs of edges in R intersect in D.[2] Let us define $c_w(G, R)$ as the smallest number of intersections in a weak realization of (G, R), $c_w(G) = \max\{c_w(G, R) : (G, R) \text{ has a weak realization}\}$, and $c_w(n) = \max\{c_w(G) : G \text{ has } n \text{ edges}\}$. Similarly define $c_r(G, R)$, $c_r(G)$, and $c_r(n)$ for realizations.[3] The quantity $cr(G) = c_w(G, \binom{E}{2})$ is known as the *crossing number* of the graph G, and was shown to have an **NP**-complete decision problem by Garey and Johnson [GJ83] in the early eighties. The other extreme case $c_w(G, \emptyset)$ is equivalent to planarity testing, and therefore in **P**.

Given a collection of curves $(C_i)_{i \in I}$ in the plane, the corresponding intersection graph is $(I, \{\{i, j\} : C_i \text{ and } C_j \text{ intersect}\}$. The *size* of a collection of curves is the number of intersection points (we assume that no three curves intersect in the same point). A graph isomorphic to the intersection graph of a collection of curves in the plane is called a *string graph*. Let $c_s(G)$ be the size of a smallest (in the sense of size defined above) set of curves whose intersection graph is isomorphic to G, and define $c_s(n) = \max\{c_s(G) : G \text{ has } n \text{ edges}\}$.

The follwing relationships between the functions we defined are well known:

[2] Kratochvíl [Kra98, Kra91a, Kra91b] calls (G, R) am *abstract topological graph*, and uses the word *feasible* for weakly realizable.

[3] The function definitions here and in the following paragraph are based on similar ones in the papers by Kratochvíl and Matoušek [KM91, Kra98].

(i) $c_w(n) \leq c_r(n)$,

(ii) $c_r(n) \leq c_s(3n^2)$, and

(iii) $c_s(n) \leq c_w(n)$.

The first inequality follows from $c_w(G,R) \leq \max\{c_r(G,R') : R' \subseteq R,$ and (G,R) has a realization$\}$, the second from Kratchovíls reduction of realizability to string graphs [Kra91b, Proposition 1], and the third from his reduction of string graphs to weak realizability [Kra91b, Proposition 5].

Therefore c_w, c_r, and c_s are polynomially related. Kratchovíl and Matoušek [KM91] showed that $c_w(n) \geq 2^{cn}$ for some positive constant c. Our main result shows that $c_w(n) \leq n2^n$ (Kratchovíl and Matoušek conjectured an upper bound of 2^{n^k}). This implies the decidability of string graphs, which was a long-standing open problem in the field.

3 Bounding the Number of Intersections

If we assign each curve in a collection of curves a unique number, we can look at the intersections of the curves along a particular curve as a word over an alphabet. The basic idea of the proof is to show that if such a word is too long, it contains a substructure which allows a drawing of the collection of curves of smaller size (lesser number of intersections). Hence we can bound the number of intersections along each curve, thereby bounding the size of the whole drawing.

Lemma 3.1 *Every word of length $\geq 2^n$ over an alphabet of size n contains a non-trivial subword in which each character occurs an even number of times.*

Proof Let $\Sigma = \{1, \ldots, n\}$, and $w \in \Sigma^*, |w| \geq 2^n$. To every $i \in \{0, ..., 2^n\}$ assign a vector v_i in $\mathbb{Z}_2^n$ whose j-th coordinate is the parity of the number of occurrences of the symbol j in the prefix of w of length i. (In particular v_0 is the all-zero vector.) Since there are $2^n + 1$ indizes, but only 2^n vectors in $\mathbb{Z}_2^n$, there are $0 \leq i < j \leq 2^n$ such that $v_i = v_j$. The non-trivial subword of w starting in position $i+1$ and ending in position j fulfills the conditions of the lemma. $\square$

Theorem 3.2 *Let G be a graph with m edges, $R \subseteq \binom{E}{2}$ such that (G,R) is weakly realizable, and let D be a weak realization of (G,R) with the minimal number of intersections. Then for any edge $e \in G$ there are less than 2^m intersections on the curve realizing e in D.*

Proof Suppose not. Let D be a weak realization of (G,R) with the minimal number of intersections and let e be an edge of G which has more than $2^m - 1$ intersections in D. By Lemma 3.1 we can choose a non-trivial segment of this edge which is intersected an even number of times by any other edge. Draw a window around this part which contains no other intersections. For each edge f assign numbers $1, 2, \ldots, 4n_f$ ($n_f \in \mathbb{N}$) to intersections with the window in the order they appear on f (choose an arbitrary orientation of f). For an example see Figure 2.

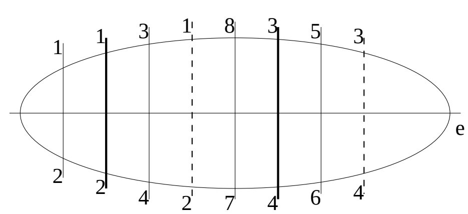

Figure 2: Part of e with surrounding window.

We can assume that the window is a circle, that e within the window is a straight line passing through the center, and that for every f intersections $2i-1$ and $2i$ are mirror images of each other (with e as the mirror), $i = \{1, \ldots, 2n_f\}$.

Clear the inside of the window with the exception of e. For each edge f there is connection between intersection $4i-2$ and $4i-1$ lying completely outside the window, $i \in \{1, \ldots, n_f\}$. Look at all of these connections and use circular inversion to bring them inside the circle. Now mirror everything inside the window along e.

This yields for every edge f a connection between $4i-3$ and $4i$, $i \in \{1, \ldots, n_f\}$, inside the window. Since $4n_f$ is the last intersection of f with the window we have that the edge f still connects its endpoints (here we needed that f intersects e an even number of times). Note that we have reduced the number of intersections of f with the window from $4n_f$ to $2n_f$. Every intersection between curves inside the circle corresponds to an intersection outside and hence this new realization respects R. We now move the part of e in the circle to coincide with one of the two arcs into which e separates the full circle. We choose the one which results in the smaller number of intersections with e. Since each edge f causes at most $2n_f$ intersections with the window, this means that the number of intersections on e within the area of the window has been halved, and hence the total number of intersections of the drawing has been decreased, a contradiction. $\square$

Corollary 3.3 *String graph recognition is in* **NEXP**.

Proof Theorem 3.2, and the fact that $c_s(n) \leq c_w(n)$ (see the preliminaries) shows that if G is a string graph, there is a collection of curves of size at most $n2^n$ whose intersection graph is isomorphic to G. We can consider the drawing of the collection of curves as a planar graph (each intersection point becoming a vertex) with at most $n2^n$ vertices. By a result of Schnyder [Sch90], and de Fraysseix, Pach, and Pollack [dFPP90] there is a drawing of this graph on a planar straight-line grid of at most $(n2^n)^2$ vertices. Hence in **NEXP** we can guess a graph on such a grid and verify whether its intersection graph is isomorphic to G. $\square$

The same argument shows that we can decide the (weak) realizability of a topological graph (G, R) in **NEXP**.

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