A Duality Theory for Bilattices

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July 9, 1999

Abstract

Recent studies of the algebraic properties of bilattices have provided insight into their internal structures, and have led to practical results, especially in reducing the computational complexity of bilattice-based multi-valued logic programs. In this paper the representation theorem for interlaced bilattices with negation found in [19] and extended to arbitrary interlaced bilattices without negation in [2] is presented. A natural equivalence is then established between the category of interlaced bilattices and the cartesian square of the category of bounded lattices. As a consequence a dual natural equivalence is obtained between the category of distributive bilattices and the coproduct of the category of bounded Priestley spaces with itself. Some applications of these equivalences are given. The subdirectly irreducible interlaced bilattices are characterized in terms of subdirectly irreducible lattices. A known characterization of the join-irreducible elements of the "knowledge" lattice of an interlaced bilattice is used to establish a natural equivalence between the category of finite, distributive bilattices and the category of posets of the form $\mathbf{P} \oplus_{\perp} \mathbf{Q}$.

Introduction

Bilattices are algebras with two separate lattice structures. They have been used as the basis for a denotational semantics for systems of inference that arise in artificial intelligence and knowledge-based logic programming ([7, 9]). In particular, they have been used to provide a general framework for an efficient procedural semantics of logic programming languages that can deal with incomplete as well as contradictory information. Such systems must

[0]*Keywords:* representation of bilattice, category of bilattices, logic programming, knowledge, Priestly duality, multi-valued logic, subdirectly irreducibles, join irreducibles.
1991 AMS Subject Classification: Primary: 06B15, *Secondary:* 08A70, 18A23, 68N17.

have two common characteristics: first they must rely on the expressive power of an underlying multi-valued logic, and secondly, they should be able to interpret statements not only based on their truth or falsity, but also based on some measure of the knowledge or information contained within those statements. In such semantics the state of an agent's knowledge about a possible event plays as an important a role as the event's truth value. In this context the two lattice orderings of a bilattice are viewed as representing, respectively, the relative degrees of truth and knowledge of possible events.

Originally, Ginsberg [9] suggested using bilattices as the underlying framework for various AI inference systems including those based on default logics, truth maintenance systems, probabilistic logics, and others. These ideas were later pursued [7, 8, 13] in the context of logic programming semantics. Also recently, bilattices and their extensions have been used in the literature to model a variety of reasoning mechanisms about uncertainty in the presence of incomplete or contradictory information. For example, in [18], a variant of Fitting's extension of logic programming to bilattices was used to deal with a form of negation as failure as well as a second explicit negation in logic programs. In [11] bilattices were extended to include a third ordering (called the *precision* ordering) in order to effectively deal with varying degrees of belief and doubt in probabilistic deductive databases.

Studies of the algebraic structure of bilattices have led to practical results, particularly in reducing the computational complexity of bilattice-based logic programs. For example, in [13] it was shown that for finite distributive bilattices (and, more generally, bilattices with the *descending chain property*), we can restrict our attention to derivations that range over a relatively small subset of special truth-values. These special truth values turn out to be the *join irreducible* elements of the knowledge part of the bilattice, and they provide the basis for an efficient procedural semantics for multivalued logic programs. Ginsberg [9] has discussed the ramifications of reducing the complexity of bilattice based inference systems by focusing on a smaller set of representative elements called *grounded* elements. Join-irreducible elements provide an even smaller set of representative elements which represent the most "primitive" bits of information. In fact, this difference could be exponential for certain classes of bilattices. More recently, the notion of join-irreduciblity has been used in connection with a proof theory for bilattice-based logics [1]. It is therefore important to further study bilattices and their algebraic properties. This, in part, is the motivation behind the present work.

A bilattice has *negation* if there is a bijection of order 2 with the property that it preserves one of the orderings and inverts the other. A bilattice is *interlaced* if the join and meet operations of each of its two lattice orderings is monotonic with respect to the other ordering. In this paper we extend the representation theorem for interlaced bilattices with negation found in [19] to arbitrary interlaced bilattices. We then refine it to construct a natural equivalence between the category of interlaced bilattices and the product (in the category of categories) of the category of ordinary bounded lattices with itself. This natural equivalence induces another equivalence between the category of distributive bilattices and the product of the category of bounded distributive lattices with itself. This in turn gives rise to a dual natural equivalence between the category of distributive bilattices and the coproduct of the category of bounded Priestley spaces with itself. In analogy with distributive lattices, this specializes to a duality between the categories of finite distributive bilattices and the coproduct of the category of finite ordered sets (posets) with itself.

In the case of interlaced bilattices with negation, we obtain the same pattern of equivalences but with the product or coproduct of each category replaced by the category itself. For example, the category of interlaced bilattices with negation is naturally equivalent to the category of bounded lattices.

These equivalences give considerable insight into the structure of join-irreducible elements of an interlaced bilattice. This structure is investigated in the last part of the paper. The representation of a finite, distributive bilattice in terms of the poset of join-irreducibles of its knowledge lattice has proved useful in constructing simplified operational semantics for knowledge-based logic programs based on bilattices; see [13].

Basic theory of bilattices

Definition 1 A *bilattice* is an algebra $\mathbf{B} = \langle B, \wedge_1, \vee_1, 0_1, 1_1, \wedge_2, \vee_2, 0_2, 1_2 \rangle$ such that $\mathbf{B}_1 = \langle B, \wedge_1, \vee_1, 0_1, 1_1 \rangle$ and $\mathbf{B}_2 = \langle B, \wedge_2, \vee_2, 0_2, 1_2 \rangle$ are bounded lattices.

By a *negation* on $\mathbf{B}$ we mean a unary operation $\neg$ on B satisfying the conditions

1. $\neg\neg x = x$,
2. $\neg(x \vee_1 y) = \neg x \wedge_1 \neg y, \quad \neg(x \wedge_1 y) = \neg x \vee_1 \neg y,$
3. $\neg(x \vee_2 y) = \neg x \vee_2 \neg y, \quad \neg(x \wedge_2 y) = \neg x \wedge_2 \neg y.$

Although the terminology for bilattices is not uniform, it has become more-or-less standard to use the term *bilattice* to refer to what we call a bilattice with negation. For this reason, we often speak of a bilattice *without negation* for emphasis.

The lattice ordering corresponding to the bounded lattice $\mathbf{B}_1$ will be denoted by $\leq_1$ and the lattice ordering corresponding to $\mathbf{B}_2$ by $\leq_2$; often the bilattice $\mathbf{B}$ is written in the form $\langle B, \leq_1, \leq_2 \rangle$. Alternatively, $\leq_1$ and $\leq_2$ are often denoted by $\leq_{\mathrm{t}}$ and $\leq_{\mathrm{k}}$, respectively, reflecting the fact that they represent the "truth" and "knowledge" orderings when the bilattice serves as the basis for the denotational semantics of a knowledge-based logic program.

Any single lattice $\mathbf{L} = \langle L, \leq \rangle$ determines two bilattices $\mathbf{L}^+ = \langle L, \leq, \leq \rangle$ and $\mathbf{L}^- = \langle L, \geq, \leq \rangle$ in a natural way. Alternatively, if $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$, then $\mathbf{L}^+ = \langle L, \wedge, \vee, 0, 1, \wedge, \vee, 0, 1 \rangle$ and $\mathbf{L}^- = \langle L, \vee, \wedge, 1, 0, \wedge, \vee, 0, 1 \rangle$. The two simplest examples of nontrivial bilattices without negation are the bilattices $\mathcal{TWO}^+$ and $\mathcal{TWO}^-$, where $\mathcal{TWO}$ is the $2-$element lattice. The simplest example of a nontrivial bilattice with negation is the bilattice $\mathcal{FOUR}$, depicted in Figure 1; it is the algebraic representation of the four-valued logic of Belnap [3]. There are many other interesting nonclassical logics that are useful for knowledge-based logic programming and that can be represented using bilattices. For a detailed discussion see [7, 9].

Note that, if $\mathbf{B}$ has negation, then $x \leq_1 y$ if and only if $\neg y \leq_1 \neg x$ and $x \leq_2 y$ if and only if $\neg x \leq_2 \neg y$. (The latter equivalence reflects the intuition that, if an agent has more knowledge of one particular possible event E_1 than of another event E_2, it also has more knowledge of the logical negation of E_1 than of the logical negation of E_2.) Moreover, $\neg : \langle B, \wedge_1, \vee_1 \rangle \to \langle B, \vee_1, \wedge_1 \rangle$ and $\neg : \langle B, \wedge_2, \vee_2 \rangle \to \langle B, \wedge_2, \vee_2 \rangle$ are lattice isomorphisms. This immediately implies that $\neg 0_1 = 1_1, \neg 1_1 = 0_1$ and also that 0_2 and 1_2 remain fixed under negation.

Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be two bilattices. As usual, a mapping $f : B_1 \to B_2$ is called a *bilattice homomorphism* if it preserves all bilattice operations (including negation if it is present). The collection of all bilattices together with all bilattice homomorphisms forms a category, called the *category of bilattices* and denoted by $\mathcal{BL}$. The category of bilattices with negation is denoted by $\mathcal{BL}_\neg$.

Definition 2 A bilattice $\mathbf{B}$ (with or without negation) is called *interlaced* if each of $\wedge_1, \vee_1, \wedge_2, \vee_2$ is monotonic with respect to both orderings $\leq_1$ and $\leq_2$. $\mathbf{B}$ is called *distributive* if for every $\diamond, \Box \in \{\wedge_1, \vee_1, \wedge_2, \vee_2\}$ and all

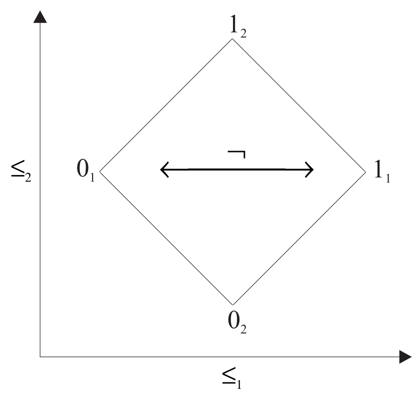

Figure 1: The bilattice $\mathcal{FOUR}$

$x, y, z \in B$, $x \diamond (y \,\square\, z) = (x \diamond y) \,\square\, (x \diamond z)$.

The term *interlaced bilattice* was introduced in [7]. It is worth pointing out that interlaced bilattices form a variety that can be axiomatized by the following four identities:

$$((x \wedge_1 y) \wedge_2 z) \wedge_1 (y \wedge_2 z) = (x \wedge_1 y) \wedge_2 z,$$

$$((x \wedge_2 y) \wedge_1 z) \wedge_2 (y \wedge_1 z) = (x \wedge_2 y) \wedge_1 z,$$

$$((x \wedge_1 y) \vee_2 z) \wedge_1 (y \vee_2 z) = (x \wedge_1 y) \vee_2 z,$$

$$((x \vee_2 y) \wedge_1 z) \vee_2 (y \wedge_1 z) = (x \vee_2 y) \wedge_1 z.$$

In [19] the term Padmanabhan bilattice is used instead of the term interlaced bilattice for those bilattices that satisfy these four identities.

The bilattices $\mathbf{L}^+$ and $\mathbf{L}^-$ are interlaced for every lattice $\mathbf{L}$. We also remark that every distributive bilattice is interlaced.

The full subcategories of $\mathcal{BL}$ having as their objects all interlaced and distributive bilattices will be denoted, respectively, by $\mathcal{IBL}$ and $\mathcal{DBL}$. The analogous subcategories of $\mathcal{BL}_\neg$ are denoted, respectively, by $\mathcal{IBL}_\neg$ and $\mathcal{DBL}_\neg$.

Definition 3 Let $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ and $\mathbf{L}' = \langle L', \wedge', \vee', 0', 1'\rangle$ be bounded lattices. Define $\mathcal{B}(\mathbf{L}, \mathbf{L}') = \langle L \times L', \sqcap_1, \sqcup_1, \perp_1, \top_1, \sqcap_2, \sqcup_2, \perp_2, \top_2\rangle$ as follows: for all $(x, x'), (y, y') \in L \times L'$,

$$(x, x') \sqcap_1 (y, y') = (x \wedge y,\, x' \vee' y'), \quad (x, x') \sqcup_1 (y, y') = (x \vee y,\, x' \wedge' y'),$$

$$(x, x') \sqcap_2 (y, y') = (x \wedge y,\, x' \wedge' y'), \quad (x, x') \sqcup_2 (y, y') = (x \vee y,\, x' \vee' y'),$$

$$\perp_1 = (0, 1'), \quad \top_1 = (1, 0'), \quad \perp_2 = (0, 0'), \quad \top_2 = (1, 1').$$

If $h\colon \mathbf{L} \cong \mathbf{L}'$ is a lattice isomorphism, we define

$$\sim (x, x') = (h^{-1}(x'), h(x)).$$

$\mathcal{B}(\mathbf{L}, \mathbf{L}')$ with the operation $\sim$ adjoined is denoted by $\mathcal{B}_h(\mathbf{L}, \mathbf{L}')$. For any bounded lattice $\mathbf{L}$ we write $\mathcal{B}(\mathbf{L})$ for $\mathcal{B}_h(\mathbf{L}, \mathbf{L})$, where h is the identity automorphism on $\mathbf{L}$.

$\mathcal{B}(\mathbf{L}, \mathbf{L}')$ is called the *product bilattice associated with* $\mathbf{L}$ and $\mathbf{L}'$. $\mathcal{B}(\mathbf{L})$ is the *square bilattice with negation associated with* $\mathbf{L}$.

The product bilattice with negation associated with the two-element lattice is $\mathcal{FOUR}$.

Theorem 4 ([7, 19]) 1. *For any pair of bounded lattices* $\mathbf{L}$ *and* $\mathbf{L}'$, $\mathcal{B}(\mathbf{L}, \mathbf{L}')$ *is an interlaced bilattice. Moreover,* $\mathcal{B}(\mathbf{L}, \mathbf{L}')$ *is distributive if and only if* $\mathbf{L}$ *and* $\mathbf{L}'$ *are both distributive.*

2. *For any bounded lattice* $\mathbf{L}$, $\mathcal{B}(\mathbf{L})$ *is an interlaced bilattice with negation. Moreover,* $\mathcal{B}(\mathbf{L})$ *is distributive if and only if* $\mathbf{L}$ *is distributive.*

The following representation theorem is the key to all the categorical equivalences described in the introduction.

Theorem 5 1. *For every interlaced bilattice* $\mathbf{B}$, *there exists a pair* $\mathbf{L}$, $\mathbf{L}'$ *of bounded lattices such that* $\mathbf{B} \cong \mathcal{B}(\mathbf{L}, \mathbf{L}')$.

2. *For every interlaced bilattice* $\mathbf{B}$ *with negation, there exists a bounded lattice* $\mathbf{L}$ *such that* $\mathbf{B} \cong \mathcal{B}(\mathbf{L})$.

This theorem has a complicated evolution. It was formulated for distributive bilattices without negation, and the essential ideas of the proof presented, in [9] and [7]. A complete proof for the distributive case with negation can be found in [13]. Independently, the theorem for interlaced bilattices with negation was proved in [19]. Although the theorem is formulated in [19] for bilattices with negation, an analysis of the proof shows that the assumption of negation is not used in any essential way. The proof uses some general results on quasilattices and the Płonka sums of lattices over a semilattice. We sketch here a simpler and more direct proof along the lines of the proof in [13][1].

For the purpose of the proof, the following definition due to Ginsberg [9] is useful. In the present context we find it more suitable to use the terms positive and negative for what in [9] are referred to as t-grounded and f-grounded elements, respectively.

Definition 6 Let $\mathbf{B} = \langle B, \leq_1, \leq_2 \rangle$ be an interlaced bilattice. An element $x \in B$ is called *positive* if, for every $y \in B$, $x \leq_1 y$ implies $x \leq_2 y$. It is called *negative* if, for every $y \in B$, $y \leq_1 x$ implies $x \leq_2 y$. Denote by $\mathrm{POS}(\mathbf{B})$ and $\mathrm{NEG}(\mathbf{B})$ the sets of positive and negative elements, respectively, of $\mathbf{B}$.

Lemma 7 *Let* $\mathbf{B}$ *be an interlaced bilattice.*

1. $\mathrm{POS}(\mathbf{B}) = [0_2, 1_1]_{\leq_1} = [0_2, 1_1]_{\leq_2}$;
2. $\mathrm{NEG}(\mathbf{B}) = [0_2, 0_1]_{\geq_1} = [0_2, 0_1]_{\leq_2}$.

Proof. 1. Assume $x \in [0_2, 1_1]_{\leq_1}$ and $x \leq_1 y$. Then $0_2 \leq_1 x \leq_1 y$. Thus $0_2 \vee_1 x = x$ and $y \vee_1 x = y$. From $0_2 \leq_2 y$ and the monotonicity of $\vee_1$ with respect to $\leq_2$ we have $0_2 \vee_1 x \leq_2 y \vee_1 x$, i.e., $x \leq_2 y$. So $x \in \mathrm{POS}(\mathbf{B})$. Assume now that x is positive. Then from $x \leq_1 1_1$ we get $0_2 \leq_2 x \leq_2 1_1$, i.e., $x \in [0_2, 1_1]_{\leq_2}$. Finally, assume $x \in [0_2, 1_1]_{\leq_2}$. From $x \leq_2 1_1$ we get $1_1 \wedge_2 x = x$ and from $0_2 \leq_1 1_1$ we have $0_2 \wedge_2 x \leq_1 1_1 \wedge_2 x$. So $0_2 \leq_1 x$, i.e., $x \in [0_2, 1_1]_{\leq_1}$. Hence $[0_2, 1_1]_{\leq_1} \subseteq \mathrm{POS}(\mathbf{B}) \subseteq [0_2, 1_1]_{\leq_2} \subseteq [0_2, 1_1]_{\leq_1}$.

This establishes the first part of the lemma. The second part is established by a similar argument. ■

As a consequence of Part 1 of this lemma, $\langle [0_2, 1_1]_{\leq_1}, \leq_1 \rangle$ and $\langle [0_2, 1_1]_{\leq_2}, \leq_2 \rangle$ are identical lattices. We denote this lattice by $\mathbf{POS}(\mathbf{B})$. Similarly, we set

[1] We thank the referree who pointed out that this proof was first given by Avron [2] and later strengthened by Pynko [17] in an unpublished paper. Both, in turn, were not aware of the proof of Romanowska and Trakul [19]. Although our proof is essentially the same as the one presented in [2], we leave it in the text for the sake of completeness. Avron also obtained some results on the equational bases of the variety of interlaced bilattices inspired by some similar results that had previously been obtained by Jónsson [10] for the variety of distributive bilattices.

$\mathbf{NEG}(\mathbf{B}) = \langle [0_2, 0_1]_{\geq_1}, \geq_1\rangle = \langle [0_2, 0_1]_{\leq_2}, \leq_2\rangle$. In case $\mathbf{B}$ has negation, $\neg$ (when restricted to $\mathrm{POS}(\mathbf{B})$) is an isomorphism between $\mathbf{POS}(\mathbf{B})$ and $\mathbf{NEG}(\mathbf{B})$; this is easy to check. We note that $\mathbf{POS}(\mathbf{B})^+ = \langle \mathrm{POS}(\mathbf{B}), \leq_2, \leq_2\rangle$ and $\mathbf{NEG}(\mathbf{B})^- = \langle \mathrm{NEG}(\mathbf{B}), \geq_2, \leq_2\rangle$.

As a corollary of the lemma we have the following characterization of positive and negative elements of an interlaced bilattice in terms of its representation as the product bilattice associated with bounded lattices $\mathbf{L}, \mathbf{L}'$. This result was obtained for distributive bilattices in [13].

Corollary 8 *Let $\mathbf{L}$ and $\mathbf{L}'$ be bounded lattices, and let 0 and $0'$ be the respective least elements. An element x of $\mathcal{B}(\mathbf{L}, \mathbf{L}')$ is positive if and only if $x = (y, 0')$, for some $y \in L$, and it is negative if and only if $x = (0, y')$, for some $y' \in L'$.*

Proof. Let $\mathcal{B}(\mathbf{L}, \mathbf{L}') = \langle L \times L', \sqsubseteq_1, \sqsubseteq_2\rangle$. It is easy to check that $[\bot_2, \top_1]_{\sqsubseteq_2} = L \times \{0'\}$ and $[\bot_2, \bot_1]_{\sqsubseteq_2} = \{0\} \times L'$. ∎

Proof of Theorem 5:

Part 1. We show that $\mathbf{B} \cong \mathcal{B}(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$. Let

$\mathcal{B}(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B})) = \langle \mathrm{POS}(\mathbf{B}) \times \mathrm{NEG}(\mathbf{B}), \sqcap_1, \sqcup_1, \bot_1, \top_1, \sqcap_2, \sqcup_2, \bot_2, \top_2\rangle$.

Then, for all $(x, x'), (y, y') \in \mathrm{POS}(\mathbf{B}) \times \mathrm{NEG}(\mathbf{B})$,

$$
\begin{array}{rcccl}
(x, x') \sqcap_1 (y, y') & = & (x \wedge_2 y,\, x' \vee_2 y') & = & (x \wedge_1 y,\, x' \wedge_1 y'),\\
(x, x') \sqcup_1 (y, y') & = & (x \vee_2 y,\, x' \wedge_2 y') & = & (x \vee_1 y,\, x' \vee_1 y'),\\
(x, x') \sqcap_2 (y, y') & = & (x \wedge_2 y,\, x' \wedge_2 y') & = & (x \wedge_1 y,\, x' \vee_1 y'),\\
(x, x') \sqcup_2 (y, y') & = & (x \vee_2 y,\, x' \vee_2 y') & = & (x \vee_1 y,\, x' \wedge_1 y').
\end{array}
$$

$$\bot_1 = (0_2, 0_1), \quad \top_1 = (1_1, 0_2), \quad \bot_2 = (0_2, 0_2), \quad \top_2 = (1_1, 0_1).$$

We note for future reference that

$$\mathcal{B}(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B})) = \mathbf{POS}(\mathbf{B})^+ \times \mathbf{NEG}(\mathbf{B})^-,$$

where "$\times$" denotes the ordinary direct product of bilattices.

For each $x \in B$ we define

$$f(x) = (x \wedge_2 1_1,\, x \wedge_2 0_1).$$

By Lemma 7, $f\colon B \to \mathrm{POS}(\mathbf{B}) \times \mathrm{NEG}(\mathbf{B})$. We will show that f is an isomorphism between $\mathbf{B}$ and $\mathcal{B}(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$. For this purpose we first prove that, for any $x \in B$,

$$x = (x \wedge_2 1_1) \vee_2 (x \wedge_2 0_1). \tag{1}$$

Trivially $x \leq_1 1_1$. Thus by the interlacing assumption, $x \leq_1 x \wedge_2 1_1$. Again applying the interlacing assumption, we get $x = x \vee_2 (x \wedge_2 0_1) \leq_1 (x \wedge_2 1_1) \vee_2 (x \wedge_2 0_1)$. Trivially $0_1 \leq_1 x$. Thus, $x \wedge_2 0_1 \leq_1 x$. Hence $(x \wedge_2 1_1) \vee_2 (x \wedge_2 0_1) \leq_1 (x \wedge_2 1_1) \vee_2 x = x$. This establishes (1).

It follows immediately from (1) that f is injective. To see it is surjective, consider any $(x, y) \in \mathrm{POS}(\mathbf{B}) \times \mathrm{NEG}(\mathbf{B})$. Note that $y \leq_1 0_2 \leq_1 x$. So $x \vee_2 y \leq_1 x \vee_2 x = x$, and hence $(x \vee_2 y) \wedge_2 1_1 \leq_1 x \wedge_2 1_1 = x$; the last equality holds because x is positive, and thus $x \leq_2 1_1$. On the other hand, from $x \leq_1 1_1$ we also get $x = (x \vee_2 y) \wedge_2 x \leq_1 (x \vee_2 y) \wedge_2 1_1$. Thus we have $(x \vee_2 y) \wedge_2 1_1 = x$, and in a similar way we can show $(x \vee_2 y) \wedge_2 0_1 = y$. So $f(x \vee_2 y) = (x, y)$, and hence f is surjective. We have shown that f is a bijection between the two bilattices $\mathbf{B}$ and $\mathcal{B}(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$.

In order to prove it is a bilattice isomorphism, it suffices to show that it preserves the two lattice orderings. From the construction of $\mathcal{B}(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$ it follows that, for $(x, x'), (y, y') \in \mathrm{POS}(\mathbf{B}) \times \mathrm{NEG}(\mathbf{B})$, $(x, x') \sqsubseteq_1 (y, y')$ if and only if $x \leq_1 y$ and $x' \leq_1 y'$, and $(x, x') \sqsubseteq_2 (y, y')$ if and only if $x \leq_2 y$ and $x' \leq_2 y'$. Consider any $x, y \in B$. Then $x \leq_1 y$ implies (by the interlacing condition) $(x \wedge_2 1_1) \leq_1 (y \wedge_2 1_1)$ and $(x \wedge_2 0_1) \leq_1 (y \wedge_2 0_1)$, which in turn is equivalent to $f(x) \sqsubseteq_1 f(y)$. Similarly, $x \leq_2 y$ implies $(x \wedge_2 1_1) \leq_2 (y \wedge_2 1_1)$ and $(x \wedge_2 0_1) \leq_2 (y \wedge_2 0_1)$, and hence $f(x) \sqsubseteq_2 f(y)$.

This completes the proof of Part 1.

Part 2. Assume now that $\mathbf{B}$ has a negation $\neg$. We show that $\mathbf{B} \cong \mathcal{B}(\mathbf{POS}(\mathbf{B}))$ ($\cong \mathcal{B}(\mathbf{NEG}(\mathbf{B}))$). Recall the $\neg$ (restricted to $\mathrm{POS}(\mathbf{B})$) is an isomorphism between the lattices $\mathbf{POS}(\mathbf{B})$ and $\mathbf{NEG}(\mathbf{B})$. Let $\sim$ denote the negation of $\mathcal{B}_\neg(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$. Then, for all $(x, x') \in \mathrm{POS}(\mathbf{B}) \times \mathrm{NEG}(\mathbf{B})$,

$$\sim (x, x') = (\neg x', \neg x).$$

It is easy to check that the mapping $(a, b) \mapsto (a, \neg b)$ is an isomorphism between the two bilattices with negation $\mathcal{B}_\neg(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$ and $\mathcal{B}(\mathbf{POS}(\mathbf{B}))$. Consequently, in order to prove the theorem it is sufficient to show that $\mathbf{B} \cong \mathcal{B}_\neg(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$; to show this it only remains to verify that f preserves negation.

$$\begin{aligned}
\sim f(x) &= \sim (x \wedge_2 1_1,\ x \wedge_2 0_1) \\
&= (\neg(x \wedge_2 0_1),\ \neg(x \wedge_2 1_1)) \\
&= (\neg x \wedge_2 \neg(0_1),\ \neg x \wedge_2 \neg(1_1)) \\
&= (\neg x \wedge_2 1_1,\ \neg x \wedge_2 0_1) \\
&= f(\neg x).
\end{aligned}$$

This completes the proof of Theorem 5. ∎

Note that as a corollary of the proof we have that

$$\mathbf{B} \cong \mathbf{POS}(\mathbf{B})^{+} \times \mathbf{NEG}(\mathbf{B})^{-},$$

for every interlaced bilattice $\mathbf{B}$.

Corollary 9 1. *An algebra* $\mathbf{B}$ *is a distributive bilattice if and only if there exist bounded, distributive lattices* $\mathbf{L}$, $\mathbf{L}'$ *such that* $\mathbf{B} \cong \mathcal{B}(\mathbf{L}, \mathbf{L}')$.

2. $\mathbf{B}$ *is a distributive bilattice with negation if and only if there exists a bounded distributive lattice* $\mathbf{L}$ *such that* $\mathbf{B} \cong \mathcal{B}(\mathbf{L})$.

Categorical equivalences

In this section we show that the extraction of the pair $(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$ from the bilattice $\mathbf{B}$ is natural and consequently gives a categorical equivalence. The reader is referred to Mac Lane [12] for all unexplained categorical notation and terminology.

Let $\mathcal{L}$ be the category of bounded lattices $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ with morphisms all $\{0,1\}-$lattice homomorphisms. $\mathcal{DL}$ is the full subcategory of $\mathcal{L}$ whose objects are all bounded distributive lattices.

Given an interlaced bilattice $\mathbf{B}$ we set $\mathcal{L}^2(\mathbf{B}) = \langle \mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B})\rangle$. As shown above, $\mathbf{B} \cong \mathcal{B}(\mathcal{L}^2(\mathbf{B}))$, with $f_{\mathbf{B}} : \mathbf{B} \cong \mathcal{B}(\mathcal{L}^2(\mathbf{B}))$ given by

$$f_{\mathbf{B}}(x) = (x \wedge_2 1_1, x \wedge_2 0_1), \text{ for every } x \in B. \tag{2}$$

Moreover, given an arbitrary pair of bounded lattices $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ and $\mathbf{L}' = \langle L', \wedge', \vee', 0', 1'\rangle$, we can construct the interlaced bilattice $\mathcal{B}(\mathbf{L}, \mathbf{L}')$ and then extract the pair of bounded lattices $\mathcal{L}^2(\mathcal{B}(\mathbf{L}, \mathbf{L}'))$. It is not difficult to see, using Corollary 8, that $\langle \mathbf{L}, \mathbf{L}'\rangle \cong \mathcal{L}^2(\mathcal{B}(\mathbf{L}, \mathbf{L}'))$ in the product category $\mathcal{L} \times \mathcal{L}$, where $g_{\mathbf{L},\mathbf{L}'} : \langle \mathbf{L}, \mathbf{L}'\rangle \cong \mathcal{L}^2(\mathcal{B}(\mathbf{L}, \mathbf{L}'))$ is defined by

$$g_{\mathbf{L},\mathbf{L}'}(x, x') = ((x, 0'), (0, x')) \text{ for every } (x, x') \in L \times L'. \tag{3}$$

For our purposes the essential property of the constructions of $\mathcal{B}(\mathbf{L}, \mathbf{L}')$ and of $\mathcal{L}^2(\mathbf{B})$ is their naturalness. This is formalized as a functor F between the categories $\mathcal{L} \times \mathcal{L}$ and $\mathcal{IBL}$ and a functor G in the opposite direction.

Define $F_{\mathrm{Obj}} : \mathrm{Obj}(\mathcal{L} \times \mathcal{L}) \to \mathrm{Obj}(\mathcal{IBL})$, by

$$F_{\mathrm{Obj}}(\mathbf{L}, \mathbf{L}') = \mathcal{B}(\mathbf{L}, \mathbf{L}'), \quad \text{for every } \langle \mathbf{L}, \mathbf{L}'\rangle \in \mathrm{Obj}(\mathcal{L} \times \mathcal{L}).$$

$F_{\mathrm{Mor}} : \mathrm{Mor}(\mathcal{L}\times\mathcal{L}) \to \mathrm{Mor}(\mathcal{IBL})$ is defined as follows. For all $\langle \mathbf{L}, \mathbf{L}' \rangle, \langle \mathbf{M}, \mathbf{M}' \rangle \in \mathrm{Obj}(\mathcal{L}\times\mathcal{L})$ and $\langle h, h' \rangle : \langle \mathbf{L}, \mathbf{L}' \rangle \to \langle \mathbf{M}, \mathbf{M}' \rangle \in \mathrm{Mor}(\mathcal{L}\times\mathcal{L})$, $F_{\mathrm{Mor}}(h,h') : \mathcal{B}(\mathbf{L},\mathbf{L}') \to \mathcal{B}(\mathbf{M},\mathbf{M}') \in \mathrm{Mor}(\mathcal{IBL})$ is given by

$$F_{\mathrm{Mor}}(h,h')((x,x')) = (h(x), h'(x')), \quad \text{for every } (x,x') \in \mathcal{B}(\mathbf{L},\mathbf{L}').$$

It is not difficult to see that $F : \mathcal{L}\times\mathcal{L} \to \mathcal{IBL}$, acting as F_{Obj} on $\mathrm{Obj}(\mathcal{L}\times\mathcal{L})$ and as F_{Mor} on $\mathrm{Mor}(\mathcal{L}\times\mathcal{L})$, is a functor.

Define $G_{\mathrm{Obj}} : \mathrm{Obj}(\mathcal{IBL}) \to \mathrm{Obj}(\mathcal{L}\times\mathcal{L})$ by

$$G_{\mathrm{Obj}}(\mathbf{B}) = \mathcal{L}^2(\mathbf{B}), \quad \text{for every } \mathbf{B} \in \mathrm{Obj}(\mathcal{IBL}),$$

and $G_{\mathrm{Mor}} : \mathrm{Mor}(\mathcal{IBL}) \to \mathrm{Mor}(\mathcal{L}\times\mathcal{L})$ as follows: for every $\mathbf{B}, \mathbf{C} \in \mathrm{Obj}(\mathcal{IBL})$ and $k : \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\mathcal{IBL})$, $G_{\mathrm{Mor}}(k) : \mathcal{L}^2(\mathbf{B}) \to \mathcal{L}^2(\mathbf{C}) \in \mathrm{Mor}(\mathcal{L}\times\mathcal{L})$ is given by

$$G_{\mathrm{Mor}}(k)(x,y) = (k(x) \wedge_2 1_1, k(y) \wedge_2 0_1), \quad \text{for every } (x,y) \in \mathcal{L}^2(\mathbf{B}).$$

It turns out that $G : \mathcal{IBL} \to \mathcal{L}\times\mathcal{L}$, acting as G_{Obj} on $\mathrm{Obj}(\mathcal{IBL})$ and as G_{Mor} on $\mathrm{Mor}(\mathcal{IBL})$, is also a functor. We have the following theorem relating $F : \mathcal{L}\times\mathcal{L} \to \mathcal{IBL}$ and $G : \mathcal{IBL} \to \mathcal{L}\times\mathcal{L}$. Following [12], given a category $\mathcal{C}$, we use $I_{\mathcal{C}}$ to denote the identity functor on $\mathcal{C}$, and $\stackrel{\bullet}{\to}$ to denote natural transformations.

Theorem 10 *The categories $\mathcal{L}\times\mathcal{L}$ and $\mathcal{IBL}$ are naturally equivalent. More precisely, $f : I_{\mathcal{IBL}} \stackrel{\bullet}{\to} FG$ and $g : I_{\mathcal{L}\times\mathcal{L}} \stackrel{\bullet}{\to} GF$ are natural isomorphisms, where $f : \mathrm{Obj}(\mathcal{IBL}) \to \mathrm{Mor}(\mathcal{IBL})$ and $g : \mathrm{Obj}(\mathcal{L}\times\mathcal{L}) \to \mathrm{Mor}(\mathcal{L}\times\mathcal{L})$ are as defined in* (2) *and* (3), *respectively.*

Proof. Let $\langle h, h' \rangle : \langle \mathbf{L}, \mathbf{L}' \rangle \to \langle \mathbf{M}, \mathbf{M}' \rangle \in \mathrm{Mor}(\mathcal{L}\times\mathcal{L})$ and $(x,x') \in L \times L'$. Then

$$\begin{aligned} G(F(h,h'))(g_{\mathbf{L},\mathbf{L}'}(x,x')) &= (F(h,h')(x,0') \sqcap_2 (1,0'), F(h,h')(0,x') \sqcap_2 (0,1')) \\ &= ((h(x),h'(0')) \sqcap_2 (1,0'), (h(0),h'(x')) \sqcap_2 (0,1')) \\ &= ((h(x) \wedge 1, h'(0') \wedge' 0'), (h(0) \wedge 0, h'(x') \wedge' 1')) \\ &= ((h(x),0'),(0,h'(x'))) \\ &= g_{\mathbf{M},\mathbf{M}'}(h(x),h'(x')) \\ &= g_{\mathbf{M},\mathbf{M}'}(h,h')(x,x'), \end{aligned}$$

i.e., the following diagram commutes, as required.

$$\begin{array}{ccccc} & & g_{\mathbf{L},\mathbf{L}'} & & \\ & \langle \mathbf{L}, \mathbf{L}' \rangle & \longrightarrow & G(F(\mathbf{L},\mathbf{L}')) & \\ \langle h, h' \rangle & \downarrow & & \downarrow & G(F(h,h')) \\ & \langle \mathbf{M}, \mathbf{M}' \rangle & \longrightarrow & G(F(\mathbf{M},\mathbf{M}')) & \\ & & g_{\mathbf{M},\mathbf{M}'} & & \end{array}$$

Next, let $k : \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\mathcal{IBL})$ and $x \in B$. Then

$$\begin{array}{rcl} F(G(k))(f_{\mathbf{B}}(x)) & = & F(G(k))((x \wedge_2 1_1, x \wedge_2 0_1)) \\ & = & (k(x \wedge_2 1_1) \wedge_2 1_1, k(x \wedge_2 0_1) \wedge_2 0_1) \\ & = & (k(x) \wedge_2 1_1 \wedge_2 1_1, k(x) \wedge_2 0_1 \wedge_2 0_1) \\ & = & (k(x) \wedge_2 1_1, k(x) \wedge_2 0_1) = f_{\mathbf{C}}(k(x)), \end{array}$$

i.e., the following diagram commutes, as required.

$$\begin{tikzcd} \mathbf{B} \arrow[r, "f_{\mathbf{B}}"] \arrow[d, "k"'] & F(G(\mathbf{B})) \arrow[d, "F(G(k))"] \\ \mathbf{C} \arrow[r, "f_{\mathbf{C}}"'] & F(G(\mathbf{C})) \end{tikzcd}$$

Thus f and g are natural transformations between $I_{\mathcal{IBL}}$ and FG and between $I_{\mathcal{L}\times\mathcal{L}}$ and GF, respectively. Since $f_{\mathbf{B}} : \mathbf{B} \to F(G(\mathbf{B}))$ and $g_{\mathbf{L}\times\mathbf{L}'} : \mathbf{L}\times\mathbf{L}' \to G(F(\mathbf{L}\times\mathbf{L}'))$ are isomorphisms, f and g are natural isomorphisms. ■

Corollary 11 *The categories $\mathcal{DL} \times \mathcal{DL}$ and $\mathcal{DBL}$ are naturally equivalent.*

Proof. By Theorems 4 and 10, the restrictions of f and g to $\mathrm{Obj}(\mathcal{DBL})$ and $\mathrm{Obj}(\mathcal{DL} \times \mathcal{DL})$, respectively, are natural isomorphisms. ■

The well known natural equivalence between $\mathcal{DL}$ and the opposite category of bounded Priestley spaces $\mathcal{PS}^{\mathrm{op}}$ ([14, 15, 16], see also [6]) induces a natural equivalence between the corresponding product categories $\mathcal{DL}\times\mathcal{DL}$ and $\mathcal{PS}^{\mathrm{op}} \times \mathcal{PS}^{\mathrm{op}}$. Combined with the natural equivalence of the above theorem, this gives a Priestley-style duality theorem for interlaced bilattices.

Corollary 12 *The category of distributive bilattices $\mathcal{DBL}$ and the coproduct of the category of bounded Priestley spaces $\mathcal{PS}$ with itself are dually naturally equivalent categories.*

Proof. Let $\Phi : \mathcal{DL} \to \mathcal{PS}^{\mathrm{op}}$ and $\Psi : \mathcal{PS}^{\mathrm{op}} \to \mathcal{DL}$ be the well-known functors and $\psi : I_{\mathcal{DL}} \stackrel{\bullet}{\to} \Psi\Phi$ and $\phi : I_{\mathcal{PS}^{\mathrm{op}}} \stackrel{\bullet}{\to} \Phi\Psi$ the well-known natural isomorphisms that establish the dual natural equivalence between $\mathcal{DL}$ and $\mathcal{PS}$. Consider the composed functors $(\Phi\times\Phi)G : \mathcal{DBL} \to \mathcal{PS}^{\mathrm{op}}\times\mathcal{PS}^{\mathrm{op}}$ and $F(\Psi\times\Psi) : \mathcal{PS}^{\mathrm{op}} \times \mathcal{PS}^{\mathrm{op}} \to \mathcal{DBL}$. Let $\mathbf{B} \in \mathrm{Obj}(\mathcal{DBL})$. $\langle \psi_{\mathbf{POS(B)}}, \psi_{\mathbf{NEG(B)}} \rangle : G(\mathbf{B}) \to (\Psi \times \Psi)(\Phi \times \Phi)G(\mathbf{B})$ is an isomorphism in $\mathcal{DL} \times \mathcal{DL}$. Applying the functor F we get the isomorphism $F(\langle \psi_{\mathbf{POS(B)}}, \psi_{\mathbf{NEG(B)}} \rangle) : FG(\mathbf{B}) \to F(\Psi \times \Psi)(\Phi \times \Phi)G(\mathbf{B})$ in $\mathcal{DBL}$. Finally, composing this with the isomorphism $f_{\mathbf{B}} : \mathbf{B} \to FG(\mathbf{B})$ we get the isomorphism, in $\mathcal{DBL}$,

$$F(\langle \psi_{\mathbf{POS(B)}}, \psi_{\mathbf{NEG(B)}} \rangle) \circ f_{\mathbf{B}} : \mathbf{B} \to F(\Psi \times \Psi)(\Phi \times \Phi)G(\mathbf{B}).$$

The collection of these isomorphisms for each $\mathbf{B} \in \mathrm{Obj}(\mathcal{DBL})$ is clearly a natural isomorphism between the functors $I_{\mathcal{DBL}}$ and $F(\Psi \times \Psi)(\Phi \times \Phi)G$. In a similar way we have that

$$\{ \, (\Phi \times \Phi)(g_{\Psi(P),\Psi(P')}) \circ \langle \Psi_P, \Psi_{P'} \rangle : \langle P, P' \rangle \in \mathrm{Obj}(\mathcal{PS}^{\mathrm{op}} \times \mathcal{PS}^{\mathrm{op}}) \, \}$$

is a natural isomorphism between $I_{\mathcal{PS}^{\mathrm{op}} \times \mathcal{PS}^{\mathrm{op}}}$ and $(\Phi \times \Phi)GF(\Psi \times \Psi)$. ■

Theorem 10 and Corollaries 11 and 12 have corresponding analogues for bilattices with negation.

Theorem 13 *The categories $\mathcal{L}$ and $\mathcal{IBL}_\neg$ are naturally equivalent.*

Proof. Given an interlaced bilattice $\mathbf{B}$ with negation, set

$$\mathcal{L}(\mathbf{B}) = \mathbf{POS}(\mathbf{B}),$$

and, given a bounded lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$, let $\mathcal{B}(\mathbf{L})$, be as in Definition 3. The functor $F : \mathcal{L} \to \mathcal{IBL}_\neg$ is defined as follows. $F_{\mathrm{Obj}} : \mathrm{Obj}(\mathcal{L}) \to \mathrm{Obj}(\mathcal{IBL}_\neg)$ is given by

$$F_{\mathrm{Obj}}(\mathbf{L}) = \mathcal{B}(\mathbf{L}), \quad \text{for every } \mathbf{L} \in \mathrm{Obj}(\mathcal{L}),$$

and, for all $\mathbf{L}, \mathbf{M} \in \mathrm{Obj}(\mathcal{L})$ and $h : \mathbf{L} \to \mathbf{M} \in \mathrm{Mor}(\mathcal{L})$, $F_{\mathrm{Mor}}(h) : \mathcal{B}(\mathbf{L}) \to \mathcal{B}(\mathbf{M}) \in \mathrm{Mor}(\mathcal{IBL}_\neg)$ is given by

$$F_{\mathrm{Mor}}(h)(x, y) = (h(x), h(y)), \quad \text{for every } (x, y) \in \mathcal{B}(\mathbf{L}).$$

Furthermore, we note that F preserves surjections in the sense that, if $h : \mathbf{L} \to \mathbf{M}$ is a surjection then so is $F(h) : \mathcal{B}(\mathbf{L}) \to \mathcal{B}(\mathbf{M})$.

The functor $G : \mathcal{IBL}_\neg \to \mathcal{L}$ is defined by

$$G_{\mathrm{Obj}}(\mathbf{B}) = \mathcal{L}(\mathbf{B}), \quad \text{for every } \mathbf{B} \in \mathrm{Obj}(\mathcal{IBL}_\neg),$$

and, for every $\mathbf{B}, \mathbf{C} \in \mathrm{Obj}(\mathcal{IBL}_\neg)$ and $k : \mathbf{B} \to \mathbf{C} \in \mathrm{Mor}(\mathcal{IBL}_\neg)$,

$$G_{\mathrm{Mor}}(k)(x) = (k(x) \wedge_2 1_1, k(x) \wedge_2 0_1), \quad \text{for every } x \in \mathcal{L}(\mathbf{B}).$$

G also preserves surjections in the obvious sense.

If we then define $f_{\mathbf{B}} : \mathbf{B} \cong \mathcal{B}(\mathcal{L}(\mathbf{B}))$ by

$$f_{\mathbf{B}}(x) = (x \wedge_2 1_1, x \wedge_2 0_1), \text{ for every } x \in B,$$

and $g_{\mathbf{L}} : \mathbf{L} \cong \mathcal{L}(\mathcal{B}(\mathbf{L}))$ by

$$g_{\mathbf{L}}(x) = (x, x), \text{ for every } x \in L,$$

then, following the proof of Theorem 10, we obtain again a natural equivalence. ■

Corollary 14 *The categories $\mathcal{DL}$ and $\mathcal{DBL}_{\neg}$ are naturally equivalent.*

Corollary 15 *The category of distributive bilattices with negation $\mathcal{DBL}_{\neg}$ and the category of bounded Priestley spaces $\mathcal{PS}$ are dually naturally equivalent.*

Subdirect Irreducibility

As an easy application of Theorem 5 we obtain a simple characterization of the subdirectly irreducible interlaced bilattices, see, e.g., [4].

Theorem 16 *An interlaced bilattice $\mathbf{B}$ is subdirectly irreducible if and only if $\mathbf{POS}(\mathbf{B})$ is a subdirectly irreducible lattice and $\mathbf{NEG}(\mathbf{B})$ is trivial or vice-versa.*

Proof. Let $\mathbf{B} = \langle B, \leq_1, \leq_2 \rangle$ be a subdirectly irreducible interlaced bilattice. From the remark following the proof of Theorem 5 we have $\mathbf{B} \cong \mathbf{POS}(\mathbf{B})^{+} \times \mathbf{NEG}(\mathbf{B})^{-}$. So either $\mathbf{POS}(\mathbf{B})^{+}$ or $\mathbf{NEG}(\mathbf{B})^{-}$, and hence either $\mathbf{POS}(\mathbf{B})$ or $\mathbf{NEG}(\mathbf{B})$, must be trivial. Assume, without loss of generality, that $\mathbf{POS}(\mathbf{B})$ is trivial. Then $\mathbf{B} \cong \mathbf{NEG}(\mathbf{B})^{-}$. Therefore $\mathbf{NEG}(\mathbf{B})^{-}$ must be a subdirectly irreducible bilattice. But $\mathbf{NEG}(\mathbf{B})^{-}$ has exactly the same congruence lattice that $\mathbf{NEG}(\mathbf{B})$ has. So $\mathbf{NEG}(\mathbf{B})^{-}$ is a subdirectly irreducible bilattice if and only if $\mathbf{NEG}(\mathbf{B})$ is a subdirectly irreducible lattice, as required.

Conversely, by a similar argument, if one of $\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B})$ is subdirectly irreducible and the other is trivial, then $\mathbf{B} \cong \mathcal{B}(\mathbf{POS}(\mathbf{B}), \mathbf{NEG}(\mathbf{B}))$ is also subdirectly irreducible. ■

Corollary 17 *An interlaced bilattice* $\mathbf{B}$ *is subdirectly irreducible if and only if there exists a subdirectly irreducible lattice* $\mathbf{L}$ *such that* $\mathbf{B} \cong \mathbf{L}^+$ *or* $\mathbf{B} \cong \mathbf{L}^-$.

Corollary 18 $\mathcal{TWO}^+$ *and* $\mathcal{TWO}^-$ *are up to isomorphism the only subdirectly irreducible distributive bilattices.*

Theorem 16 and its second corollary can be reformulated for interlaced bilattices with negation. It turns out that it is more convenient to prove the analog of Theorem 16 as a corollary of the natural equivalence between the categories $\mathcal{L}$ and $\mathcal{IBL}_\neg$ established in Theorem 13. We thus obtain the following result of Romanowska and Trakul.

Theorem 19 ([19], Corollary 4.6) *An interlaced bilattice* $\mathbf{B}$ *with negation is subdirectly irreducible iff the lattice* $\mathcal{L}(\mathbf{B}) = \mathrm{POS}(\mathbf{B})$ *is subdirectly irreducible.*

Proof. Let $\mathbf{B} \in \mathrm{Obj}(\mathcal{IBL}_\neg)$ be subdirectly irreducible. Let $\mathbf{L} = \mathcal{L}(\mathbf{B})$ and assume, to the contrary, that $\mathbf{L}$ is not subdirectly irreducible, i.e., that $\mathbf{L}$ has a subdirect representation $h : \mathbf{L} \to \prod_{i\in I} \mathbf{L}_i$, where none of the compositions $p_i h : \mathbf{L} \to \mathbf{L}_i$ is a lattice isomorphism. Natural isomorphisms preserve products. Hence, by Theorem 13, the previously observed fact that the functor $F : \mathcal{L} \to \mathcal{IBL}_\neg$ preserves surjections, and the fact that direct products are preserved under natural equivalence, we have that $F(h) : \mathcal{B}(\mathbf{L}) \to \mathcal{B}(\prod_{i\in I} \mathbf{L}_i) \cong \prod_{i\in I} \mathcal{B}(\mathbf{L}_i)$ is a subdirect representation of $\mathcal{B}(\mathbf{L}) \cong \mathbf{B}$ with none of the compositions $p_i F(h) : \mathcal{B}(\mathbf{L}) \to \mathcal{B}(\mathbf{L}_i)$ a bilattice isomorphism, contrary to the subdirect irreducibility of $\mathbf{B}$.

Thus $\mathbf{B}$ subdirectly irreducible implies $\mathcal{L}(\mathbf{B})$ is subdirectly irreducible. A similar argument shows that, if $\mathbf{L}$ is subdirectly irreducible, then so is $\mathcal{B}(\mathbf{L})$. Hence $\mathcal{L}(\mathbf{B})$ subdirectly irreducible implies $\mathbf{B}$ ($\cong \mathcal{BL}(\mathbf{B})$) is subdirectly irreducible. ∎

Corollary 20 ([19], Theorem 4.7) $\mathcal{FOUR}$ *is up to isomorphism the only subdirectly irreducible distributive bilattice with negation.*

Proof. In view of the last part of Theorem 4, $\mathbf{B}$ is a subdirectly irreducible distributive bilattice with negation iff $\mathcal{L}(\mathbf{B})$ is a subdirectly irreducible distributive lattice. But the two-element lattice is the only subdirectly irreducible distributive lattice, and $\mathcal{L}(\mathbf{B})$ is the two-element lattice iff $\mathbf{B}$ is $\mathcal{FOUR}$. ∎

Join irreducible elements of interlaced bilattices

Based on our representation theorem for interlaced bilattices, we obtain, in this section, a characterization of the lattices of join-irreducible elements, with respect to the second ordering, of arbitrary interlaced bilattices.

The natural equivalence between $\mathcal{IBL}$ and $\mathcal{L} \times \mathcal{L}$ induces a natural equivalence between the categories $\mathcal{IBL}_{\mathrm{f}}$ and $\mathcal{L}_{\mathrm{f}} \times \mathcal{L}_{\mathrm{f}}$, i.e., category of finite interlaced bilattices and the cartesian square of the category of finite lattices. It also induces one between the categories $\mathcal{DBL}_{\mathrm{f}}$ and $\mathcal{DL}_{\mathrm{f}} \times \mathcal{DL}_{\mathrm{f}}$, i.e., the category of finite distributive bilattices and the cartesian square of the category of finite distributive lattices. When the latter equivalence is combined with the cartesian square of Birkhoff's [5] natural equivalence between $\mathcal{DL}_{\mathrm{f}}$ and $\mathcal{PO}_{\mathrm{f}}^{\mathrm{op}}$, where $\mathcal{PO}_{\mathrm{f}}$ is the category of finite posets and order preserving mappings, we obtain a natural equivalence between $\mathcal{DBL}_{\mathrm{f}}$ and $\mathcal{PO}_{\mathrm{f}}^{\mathrm{op}} \times \mathcal{PO}_{\mathrm{f}}^{\mathrm{op}} \cong (\mathcal{PO}_{\mathrm{f}} \sqcup \mathcal{PO}_{\mathrm{f}})^{\mathrm{op}}$. Under this equivalence a finite, distributive bilattice $\mathbf{B}$ is mapped into the poset of its $\leq_2$ $-$join-irreducible elements.

Let $\mathbf{L} = \langle L, \leq \rangle$ be a bounded lattice. Recall that an element $x \in L$ is called $\leq$*-join-irreducible*, or simply *join-irreducible* when the lattice is clear from context, if, for all $y, z \in L$, $x = y \vee z$ implies $x = y$ or $x = z$. The set of all nonzero join-irreducibles of $\mathbf{L}$ is denoted by $J(\mathbf{L})$ and the subposet of $\langle J(\mathbf{L}), \leq \rangle$ of $\mathbf{L}$ is denoted by $\mathbf{J}(\mathbf{L})$.

Let $\mathbf{B} = \langle B, \leq_1, \leq_2 \rangle$ be a bilattice. An element $x \in B$ is called *positive* (resp. *negative*) $\leq_2$*-join-irreducible* if it is positive (resp. negative) and join-irreducible with respect to the $\leq_2$-ordering. We denote by $J_2^+(\mathbf{B})$ (resp. $J_2^-(\mathbf{B})$) the set of all nonzero positive (resp. negative) $\leq_2$-join-irreducible elements of $\mathbf{B}$, and by $\mathbf{J}_2^+(\mathbf{B})$ (resp. $\mathbf{J}_2^-(\mathbf{B})$) the corresponding partially ordered set with the partial ordering inherited by $\leq_2$. Moreover, $J_2(\mathbf{B})$ denotes the set of all $\leq_2$-join-irreducible elements of $\mathbf{B}$, and $\mathbf{J}_2(\mathbf{B})$ is the corresponding poset.

Lemma 21 *Let* $\mathbf{B}$ *be an interlaced bilattice.*

1. $J_2^+(\mathbf{B}) = J(\mathbf{POS}(\mathbf{B}))$ *and* $J_2^-(\mathbf{B}) = J(\mathbf{NEG}(\mathbf{B}))$.

2. $\{J_2^+(\mathbf{B}),\ J_2^-(\mathbf{B})\}$ *is a bipartition of* $J_2(\mathbf{B}) - \{0_2\}$.

3. *If* $\mathbf{B}$ *has negation,* $\mathbf{J}_2^+(\mathbf{B}) \cong \mathbf{J}_2^-(\mathbf{B})$.

Proof. Part 1 is an immediate consequence of Lemma 7.

Suppose $x \in B$ is both positive and negative. Then by Parts 1 and 2 of Lemma 7 respectively we have $0_2 \leq_1 x$ and $0_2 \geq_1 x$, i.e., $x = 0_2$. So $J_2^+(\mathbf{B})$ and $J_2^-(\mathbf{B})$ are disjoint.

Clearly $J_2(\mathbf{B}) \supseteq J_2^+(\mathbf{B}) \cup J_2^-(\mathbf{B})$. For the opposite inclusion, assume $0_2 \neq x \notin J_2^+(\mathbf{B}) \cup J_2^-(\mathbf{B})$. By (1) we have $x = (x \wedge_2 1_1) \vee_2 (x \wedge_2 0_1)$. By Lemma 7, since x is nonpositive, $x \neq x \wedge_2 1_1$, and, since it is also nonnegative, $x \neq x \wedge_2 0_1$. So $x = (x \wedge_2 1_1) \vee_2 (x \wedge_2 0_1)$ is a proper $\vee_2$-decomposition of x, and hence $x \notin J_2(\mathbf{B})$. This gives Part 2. The last part of the lemma is an immediate consequence of the fact that the lattices $\mathbf{POS}(\mathbf{B})$ and $\mathbf{NEG}(\mathbf{B})$ are isomorphic under negation. ■

Let $\mathbf{P} = \langle P, \preceq \rangle$ and $\mathbf{Q} = \langle Q, \sqsubseteq \rangle$ be two disjoint partially ordered sets. Define the *lift* of $\mathbf{P}$, denoted by $\mathbf{P}_\perp$, by $\mathbf{P}_\perp = \langle P \cup \{0\}, \leq \rangle$, where $0 \notin P$ and $x \leq y$ in $\mathbf{P}_\perp$ if and only if $x = 0$ or $x \preceq y$ in $\mathbf{P}$. Define the *disjoint union* $\mathbf{P} \uplus \mathbf{Q} = \langle P \cup Q, \leq \rangle$ to be the partially ordered set with $x \leq y$ if and only if either $x, y \in P$ and $x \preceq y$ or $x, y \in Q$ and $x \sqsubseteq y$.

Given two partially ordered sets $\mathbf{P}$ and $\mathbf{Q}$, define the *separated sum* of $\mathbf{P}$ and $\mathbf{Q}$, denoted $\mathbf{P} \oplus_\perp \mathbf{Q}$, to be the poset $\mathbf{P} \oplus_\perp \mathbf{Q} = (\mathbf{P}' \uplus \mathbf{Q}')_\perp$, where $\mathbf{P}', \mathbf{Q}'$ are canonically determined isomorphic copies of $\mathbf{P}, \mathbf{Q}$, respectively, such that $P' \cap Q' = \emptyset$.

Finally, observe that, by Lemma 21, $\mathbf{J}_2(\mathbf{B}) = \mathbf{J}_2^+(\mathbf{B}) \oplus_\perp \mathbf{J}_2^-(\mathbf{B})$.

Theorem 22 1. *Let* $\mathbf{B}$ *be a (finite) interlaced bilattice. Then there exist (finite) posets* $\mathbf{P}, \mathbf{Q}$ *such that*

$$\mathbf{J}_2(\mathbf{B}) \cong \mathbf{P} \oplus_\perp \mathbf{Q}.$$

2. *Conversely, let* $\mathbf{P}, \mathbf{Q}$ *be finite posets. Then there is a finite interlaced bilattice* $\mathbf{B}$ *such that*

$$\mathbf{P} \oplus_\perp \mathbf{Q} \cong \mathbf{J}_2(\mathbf{B}).$$

Proof: (1). Suppose that $\mathbf{B} = \langle B, \leq_1, \leq_2 \rangle$ is a finite interlaced bilattice. We have already observed that $\mathbf{J}_2(\mathbf{B}) = \mathbf{J}_2^+(\mathbf{B}) \oplus_\perp \mathbf{J}_2^-(\mathbf{B}) = \mathbf{J}(\mathbf{POS}(\mathbf{B})) \oplus_\perp \mathbf{J}(\mathbf{NEG}(\mathbf{B}))$. Set $\mathbf{P} = \mathbf{J}(\mathbf{POS}(\mathbf{B}))$ and $\mathbf{Q} = \mathbf{J}(\mathbf{NEG}(\mathbf{B}))$. Then $\mathbf{J}_2(\mathbf{B}) \cong \mathbf{P} \oplus_\perp \mathbf{Q}$.

(2). Let $\mathbf{P}, \mathbf{Q}$ be finite posets. Let $\mathcal{O}(\mathbf{P}), \mathcal{O}(\mathbf{Q})$ be the lattice of order-ideals (i.e., down sets) of $\mathbf{P}, \mathbf{Q}$, respectively. Then $\mathbf{P} \cong \mathbf{J}(\mathcal{O}(\mathbf{P}))$ and $\mathbf{Q} \cong \mathbf{J}(\mathcal{O}(\mathbf{Q}))$. Next, we construct the bilattice $\mathbf{B} = \mathcal{B}(\mathcal{O}(\mathbf{P}), \mathcal{O}(\mathbf{Q}))$. By Corollary 8 and Lemma 21 we have $\mathbf{J}_2^+(\mathbf{B}) \cong \mathbf{P}$ and $\mathbf{J}_2^-(\mathbf{B}) \cong \mathbf{Q}$. Therefore, $\mathbf{P} \oplus_\perp \mathbf{Q} \cong \mathbf{J}_2(\mathbf{B})$. ■

The correspondence between finite distributive bilattices and pairs of finite posets, as described in Theorem 22, gives rise to a natural equivalence between the category of finite distributive bilattices and a subcategory of the category of finite posets. We omit the details.

Corollary 23 1. *Let* $\mathbf{B}$ *be a (finite) interlaced bilattice with negation. Then there exists a (finite) poset* $\mathbf{P}$ *such that*

$$\mathbf{J}_2(\mathbf{B}) \cong \mathbf{P} \oplus_\perp \mathbf{P}.$$

2. *Conversely, let* $\mathbf{P}$ *be a finite poset. Then there is a finite interlaced bilattice with negation* $\mathbf{B}$ *such that*

$$\mathbf{P} \oplus_\perp \mathbf{P} \cong \mathbf{J}_2(\mathbf{B}).$$

We note that part 2 of Theorem 22 can be extended to the infinite case if one considers completely join-irreducible elements with respect to the second ordering, as opposed to just join-irreducibles.

Acknowledgments: The authors wish to thank Professor A. Romanowska for helpful discussions and comments.

References

[1] O. Arieli and A. Avron, Reasoning with logical bilattices, *Journal of Logic, Language, and Information*, **5** (1996), pp. 25-63.

[2] A. Avron, The structure of interlaced bilattices, *Mathematical Structures in Computer Science*, **6** (1996), pp. 287-299.

[3] N. D. Belnap, Jr., A useful four-valued logic, in *Modern Uses of Multiple-Valued Logic*, J. Michael Dunn and G. Epstein editors, D. Reidel, Boston (1977), pp. 8-37.

[4] G. Birkhoff, *Subdirect Unions In Universal Algebra,* Bull. Am. Math. Soc., Vol. 50 (1944), pp. 764-768.

[5] G. Birkhoff, *Lattice Theory, Third Edition,* Colloquium Publications, Vol. 25 (1967), Amer. Math. Soc.

[6] B.A. Davey, H.A. Priestley, *Introduction to Lattices and Order,* Cambridge University Press (1990).

[7] M.C. Fitting, *Bilattices in Logic Programming,* The twentieth international symposium on multiple-valued logic, G. Epstein Editor, IEEE(1990), pp. 63-70.

[8] M. C. Fitting, Bilattices and semantics of logic programming, *Journal of Logic Programming*, **11** (1991), pp. 91-116.

[9] M.L. Ginsberg, *Multi-Valued Logics: A uniform approach to reasoning in artificial intelligence,* Computational Intelligence, Vol. 4 (1988), pp. 265-316.

[10] B. Jónsson, *Distributive Bilattices*, Vanderbilt University.

[11] L. V. S. Lakshmanan and F. Sadri, Probabilistic deductive database, in *Proceedings of 1994 International Logic Programming Symposium*, M. Bruynooghe editor, MIT Press (1994), pp. 254-268.

[12] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, New York, (1971).

[13] B. Mobasher, D. Pigozzi, G. Slutzki, *Multi-valued Logic Programming Semantics: An Algebraic Approach,* Theoretical Computer Science, Vol. 171 (1997), pp. 77-109.

[14] H.A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces,* Bull. Lond. Math. Soc., Vol. 2 (1970), pp. 186-90.

[15] H.A. Priestley, *Ordered topological spaces and the representation of distributive lattices,* Proc. Lond. Math. Soc., Vol. 24 (1972), pp. 507-30.

[16] H.A. Priestley, *Ordered Sets and Duality for Distributive Lattices,* Annals of Discrete Mathematics, Vol. 23 (1984), pp. 39-60.

[17] A.P. Pynko, *Regular Bilattices,* (unpublished)

[18] P. Ruet and F. Fages, Combining explicit negation and negation as failure via Belnap's logic, in *Proceedings of the Workshop on Uncertainty in Databases and Deductive Systems*, L. V. S. Lakshmanan editor Ithaca, New York (1994), pp. 1-12.

[19] A. Romanowska, A. Trakul, *On the structure of some bilattices,* Universal and Applied Algebra (K.Hatkowska, B. Stawski, eds.) World Scientific (1989), pp. 235-253.

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