Improved kernel size for Planar Dominating Set

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Abstract

Determining whether a parameterized problem is kernelizable has recently become one of the most interesting topics of research in the area of parameterized complexity and algorithms. Theoretically, it has been proved that a parameterized problem is kernelizable if and only if it is fixed-parameter tractable. Practically, applying a data-reduction algorithm to reduce an instance of a parameterized problem to an equivalent smaller instance (i.e., a kernel) has led to very efficient algorithms, and now goes hand-in-hand with the design of practical algorithms for solving NP-hard problems. Well-known examples of such parameterized problems include the vertex cover problem, which is kernelizable to a kernel of size bounded by $2^{k}$, and the planar dominating set problem, which is kernelizable to a kernel of size bounded by $335^{k}$. This paper further reduces the upper bound on the kernel size for the planar dominating set problem to $67^{k}$, improving significantly the $335^{k}$ upper bound on the kernel size for the problem given by Alber et al.. This result is obtained by introducing a new set of reduction and coloring rules which allows the derivation of nice combinatorial properties in the kernelized graph leading to a tighter bound on the size of the kernel. The paper shows how the new bound yields a simple and competitive algorithm for the planar dominating set problem.

Keywords. dominating set, kernel size, parameterized algorithms

1 Introduction

Many practical algorithms for NP-hard problems start by applying data reduction subroutines to the input instances of the problem. The hope is that after the data reduction phase the instance of the problem has shrunk to a moderate size. This makes the applicability of a second phase, such as a branch-and-bound phase, to the resulting instance more feasible. Alber et al. mentioned in [2] how a practical pre-processing algorithm for a variation of the dominating set problem, called the red/blue dominating set problem, resulted in breaking up input instances of the problem into much smaller instances. Abu-Khzam et al. [1], in their implementation of algorithms for the vertex cover problem, observed the following: “In many cases, reduction was so effective that it eliminated the core completely, and with it the need for decomposition and search”.

On the other hand, many applications seek solutions of very small sizes to fairly large input instances of NP-hard problems. This has been the main concern for the area of parameterized computation. A parameterized problem is a set of instances of the form $(x, k)$, where $x$ is the input instance, and $k$ is a positive integer called the parameter. A parameterized problem is said to be fixed-parameter tractable [11] if there is an algorithm that solves the problem in time $f(k)|x|^c$, where $c$ is a fixed constant and $f(k)$ is a recursive function.

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By fully taking the advantage of small parameter values, the development of efficient parameterized algorithms has provided a new approach for practically solving problems that are theoretically intractable. For example, parameterized algorithms for the NP-hard problem \textsc{vertex cover} \cite{5, 9} have found applications in biochemistry \cite{6}, and parameterized algorithms in computational logic \cite{18} have provided an effective method for solving practical instances of the \textsc{ml type-checking} problem, which is complete for the class \textsc{exptime} \cite{15}.

The notion of a parameterized problem being parameterized tractable, and of the problem having a good data reduction algorithm, turn out to be very closely related. If \((x, k)\) is an instance of a parameterized problem \(Q\), then by \textit{kernelizing} the instance \((x, k)\), we mean applying a polynomial time preprocessing algorithm on \((x, k)\) to construct another instance \((x', k')\) of \(Q\), called the \textit{kernel}, such that (1) \(k' \leq k\); (2) the kernel size \(|x'|\) of \(x'\) is bounded by a function of \(k'\); and (3) a solution for \((x, k)\) can be constructed in polynomial time from a solution for \((x', k')\). It has been proved that a parameterized problem is \textit{fixed-parameter tractable} if and only if the problem is kernelizable \cite{12}.

Designing efficient parameterized algorithms and constructing kernels of reasonable sizes have been two of the main topics of research in the area of parameterized complexity recently. More specifically, constructing a problem kernel has become one of the main components in the design of an efficient parameterized algorithm for a problem \cite{5, 7, 8, 9}, and designing efficient parameterized algorithms for a parameterized problem now goes hand-in-hand with the construction of a problem kernel of a moderate size for the problem. Two of the most celebrated problems that have been receiving a lot of attention recently from both perspectives, are the \textsc{vertex cover} and \textsc{planar dominating set} problems. After a long sequence of algorithms, the \textsc{vertex cover} problem can be solved in time \(O(1.286^k + n)\) \cite{9}. Moreover, the \textsc{vertex cover} problem enjoys a kernel of size bounded by \(2^k\), and reducing this bound further seems to be a very challenging task, since it would lead to an approximation algorithm for the problem of ratio smaller than \(2\)—a result believed by many people to be unlikely. The \textsc{planar dominating set} problem as well has undergone some extensive study which culminated in a recent algorithm solving the problem in time \(O(2^{15.13\sqrt{k}} + n^3)\) \cite{13}. Recently, and after many strenuous efforts, it was shown that the \textsc{planar dominating set} problem has a problem kernel of size 335\(k\) that is computable in \(O(n^3)\) time \cite{2}. The major importance of the result in \cite{2} is in its being the first result showing that the \textsc{planar dominating set} problem is kernelizable. The question of whether such a bound on the problem kernel could be significantly improved remained open.

In this paper we investigate the problem of finding a kernel of “good” size for the \textsc{planar dominating set} problem. We improve the reduction rules proposed in \cite{2}, and introduce new rules that \textit{color} the vertices of the graph enabling us to observe many new combinatorial properties of its vertices. These properties—which become clearer after the coloring process—allow us to prove a much stronger bound on the number of vertices in the reduced graph. We show that the \textsc{planar dominating set} problem has a kernel of size 67\(k\) that is computable in \(O(n^3)\) time. This is a significant improvement over the results in \cite{2}. We show how the resulting bound on the kernel size yields a very simple algorithm for the \textsc{planar dominating set} problem that beats previous algorithms for the problem, and whose running time comes even close to some of the recently proposed algorithms.

We give some definitions and notations. A graph \(G\) is said to be \textit{planar} if \(G\) can be embedded on the plane such that no two edges in \(G\) cross. It is well-known that deciding whether a graph is planar, and constructing a planar embedding of the graph in such case, can be done in linear time \cite{16}. The number of edges in a planar graph with \(n\) vertices is bounded by \(3n - 6\) \cite{10}.

A dominating set in a graph \(G\) is a set of vertices \(D\) such that every vertex in \(G\) is either
in $D$ or adjacent to at least one vertex in $D$. The size of a dominating set $D$ is the number of vertices in $D$. A minimum dominating set of $G$ is a dominating set with the minimum size. We will denote by $\gamma(G)$ the size of a minimum dominating set in $G$. The planar dominating set problem, abbreviated PLANAR-DS henceforth, is the following. Given a planar graph $G$ and a positive integer $k$, either construct a dominating set for $G$ of size $\leq k$, or report that no such a dominating set exists. It is well-known that the PLANAR-DS problem is NP-complete [14].

In the remainder of the paper we will always assume that the graph we are dealing with is planar. We will also assume the familiarity of the reader with the basic graph-theoretic definitions and terminologies. The reader is referred to [10] for further information on these subjects.

## 2 Reduction and coloring rules

In this section we present an $O(n^3)$ time preprocessing scheme that reduces the graph $G$ to a graph $G'$, such that $\gamma(G) = \gamma(G')$, and such that given a minimum dominating set for $G'$, a minimum dominating set for $G$ can be constructed in linear time. We will color the vertices of the graph $G$ with two colors: black and white. Initially, all vertices are colored black. Informally speaking, white vertices will be those vertices that we know for sure when we color them that there exists a minimum dominating set for the graph excluding all of them. The black vertices are all other vertices. Note that it is possible for white vertices to be in some minimum dominating set, but the point is that there exists at least one minimum dominating set that excludes all white vertices. We start with the following definitions that are adopted from [2] with minor additions and modifications.

For a vertex $v$ in $G$ denote by $N(v)$ the set of neighbors of $v$, and by $N[v]$ the set $N(v) \cup \{v\}$. By removing a vertex $v$ from $G$, we mean removing $v$ and all the edges incident on $v$ from $G$. For a vertex $v$ in $G$, we partition its set of neighbors $N(v)$ into three sets: $N_1(v) = \{u \in N(v) \mid N(u) = N[v] = \emptyset\}$; $N_2(v) = \{u \in N(v) - N_1(v) \mid N(u) \cap N_1(v) = \emptyset\}$; and $N_3(v) = N(v) - (N_1(v) \cup N_2(v))$.

For two vertices $v$ and $w$ we define $N(v, w) = N(v) \cup N(w)$ and $N[v, w] = N[v] \cup N[w]$. We partition $N(v, w)$ into three sets: $N_1(v, w) = \{u \in N(v, w) \mid N(u) - N[v, w] = \emptyset\}$; $N_2(v, w) = \{u \in N(v, w) - N_1(v, w) \mid N(u) \cap N_1(v, w) = \emptyset\}$; and $N_3(v, w) = N(v, w) - (N_1(v, w) \cup N_2(v, w))$.

**Definition 2.1** Let $G = (V, E)$ be a plane graph. A region $R(v, w)$ between two vertices $v$ and $w$ is a closed subset of the plane with the following properties:

1. The boundary of $R(v, w)$ is formed by two simple paths $P_1$ and $P_2$ in $V$ which connect $v$ and $w$, and the length of each path is at most three.
2. All vertices that are strictly inside (i.e., not on the boundary) the region $R(v, w)$ are from $N(v, w)$.

For a region $R = R(v, w)$, let $V[R]$ denote the vertices in $R$, i.e.,

$$V[R] := \{u \in V \mid u \text{ sits inside or on the boundary of } R\}.$$ 

Let $V(R) = V[R] - \{v, w\}$.

**Definition 2.2** A region $R = R(v, w)$ between two vertices $v$ and $w$ is called simple if all vertices in $V(R)$ are common neighbors of both $v$ and $w$, that is, $V(R) \subseteq N(v) \cap N(w)$.

We introduce the following definitions.
Definition 2.3 A region \( R = R(v, w) \) between two vertices \( v \) and \( w \) is called *quasi-simple* if \( V[R] = V[R'] \cup R^+ \), where \( R' = R'(v, w) \) is a simple region between \( v \) and \( w \), and \( R^+ \) is a set of white vertices satisfying the following conditions:

1. Every vertex of \( R^+ \) sits strictly inside \( R' \).
2. Every vertex of \( R^+ \) is connected to \( v \) and not connected to \( w \), and is also connected to at least one vertex on the boundary of \( R' \) other than \( v \).

A vertex in \( V(R) \) is called a *simple* vertex, if it is connected to both \( v \) and \( w \), otherwise it is called *non-simple*. The set of vertices \( R^+ \), which consists of the non-simple vertices in \( V(R) \), will be referred to as \( R^+(v, w) \).

For a vertex \( u \in V(G) \), denote by \( B(u) \) the set of black vertices in \( N(u) \), and by \( W(u) \) the set of white vertices in \( N(u) \). We describe next the reduction and coloring rules to be applied to the graph \( G \). The reduction and coloring rules are applied to the graph until the application of any of them does not change the structure of the graph nor the color of any vertex in the graph. The first two reduction rules, **Rule 1** and **Rule 2**, are slight modifications of Rule 1 and Rule 2 introduced in [2]. The only difference is that in the current paper they are only applied to black vertices, and not to all the vertices as in [2].

**Rule 1 ([2]).** If \( N_3(v) \neq \emptyset \) for some black vertex \( v \), then remove the vertices in \( N_2(v) \cup N_3(v) \) from \( G \), and add a new white vertex \( v' \) and an edge \((v, v')\) to \( G \).

**Rule 2 ([2]).** If \( N_3(v, w) \neq \emptyset \) for two black vertices \( v, w \), and if \( N_3(v, w) \) cannot be dominated by a single vertex in \( N_2(v, w) \cup N_3(v, w) \), then we distinguish the following two cases.

**Case 1.** If \( N_3(v, w) \) can be dominated by a single vertex in \( \{v, w\} \) then: (1.1) if \( N_3(v, w) \subseteq N(v) \) and \( N_3(v, w) \subseteq N(w) \), remove \( N_3(v, w) \) and \( N_2(v, w) \cap N(v) \cap N(w) \) from \( G \) and add two new white vertices \( z, z' \) and the edges \((v, z), (w, z), (v, z'), (w, z')\) to \( G \); (1.2) if \( N_3(v, w) \subseteq N(v) \) and \( N_3(v, w) \subseteq N(w) \), remove \( N_3(v, w) \) and \( N_2(v, w) \cap N(v) \cap N(w) \) from \( G \) and add a new white vertex \( v' \) and the edge \((v, v')\) to \( G \); and (1.3) if \( N_3(v, w) \subseteq N(w) \) and \( N_3(v, w) \subseteq N(v) \), remove \( N_3(v, w) \) and \( N_2(v, w) \cap N(w) \) from \( G \) and add a new white vertex \( w' \) and the edge \((w, w')\) to \( G \).

**Case 2.** If \( N_3(v, w) \) cannot be dominated by a single vertex in \( \{v, w\} \), then remove \( N_2(v, w) \cup N_3(v, w) \) from \( G \) and add two new white vertices \( v', w' \) and the edges \((v, v'), (w, w')\) to \( G \).

**Rule 3.** For each black vertex \( v \) in \( G \), if there exists a black vertex \( x \in N_2(v) \cup N_3(v) \), color \( x \) white, and remove the edges between \( x \) and all other white vertices in \( G \).

**Rule 4.** For every two black vertices \( v \) and \( w \), if \( N_3(v, w) \neq \emptyset \), then for every black vertex \( x \in N_2(v, w) \cup N_3(v, w) \) that does not dominate all vertices in \( N_3(v, w) \), color \( x \) white and remove all the edges between \( x \) and the other white vertices in \( G \).

**Rule 5.** For every quasi-simple region \( R = R(v, w) \) between two vertices \( v \) and \( w \), if \( v \) is black, then for every black vertex \( x \in N_2(v, w) \cup N_3(v, w) \) strictly inside \( R \) that does not dominate all vertices in \( N_2(v, w) \cup N_3(v, w) \) strictly inside \( R \), color \( x \) white and remove all the edges between \( x \) and the other white vertices in \( G \).
Rule 6. For every two white vertices \( u \) and \( v \), if \( N(u) \subseteq N(v) \), and \( u \in N_2(w) \cup N_3(w) \) for some black vertex \( w \), then remove \( v \).

Rule 7. For every black vertex \( v \), if every vertex \( u \in W(v) \) is connected to all the vertices in \( B(v) \), then remove all the vertices in \( W(v) \) from \( G \).

Rule 8. For every two black vertices \( v \) and \( w \), let \( W(v, w) = W(v) \cap W(w) \). If \( |W(v, w)| \geq 2 \) and there is a degree-2 vertex \( u \in W(v, w) \), then remove all vertices in \( W(v, w) \) except \( u \), add a new degree-2 white vertex \( u' \), and connect \( u' \) to both \( v \) and \( w \).

A graph \( G \) is said to be reduced if every vertex in \( G \) is colored white or black, and the application of Rules 1–8 leaves the graph \( G \) unchanged. That is, the application of any of the above rules does not change the color of any vertex in \( G \), nor does it change the structure of \( G \). We have the following theorem.

Theorem 2.1 Let \( G \) be a graph with \( n \) vertices. Then in time \( O(n^3) \) we can construct a graph \( G' \) from \( G \) such that: (1) \( G' \) is reduced, (2) \( \gamma(G') = \gamma(G) \), (3) there exists a minimum dominating set for \( G' \) that excludes all white vertices of \( G' \), and (4) from a minimum dominating set for \( G' \) a minimum dominating set for \( G \) can be constructed in linear time.

Proof. Given a graph \( G \), we first color all its vertices black. We then apply Rule 1 – Rule 8 given above until the application of any of these rules leaves \( G \) unchanged. Let \( G' \) be the resulting graph. Then \( G' \) is reduced by the definition of a reduced graph. Alber et al. [2] noted that each successful application of Rule 1 and Rule 2 (i.e., an application that changes the structure of the graph \( G \)) reduces the number of vertices in the graph by at least one. Hence, the total number of applications of these two rules is bounded by \( n \). By looking at Rule 3 – Rule 7, it is easy to see that each of these rules either reduces the number of vertices in \( G \) by at least one, or changes the color of at least one black vertex to white without adding any new vertices to the graph. Moreover, none of Rules 1 – Rule 7 increases the number of edges in the graph. If we look at Rule 8, it is not difficult to see that each successful application of this rule reduces the number of edges in the graph by at least 1. Noting that the number of edges in a planar graph is linear in the number of vertices, and that the application of the rules become unnecessary if the graph does not contain any black vertices, we conclude that the total number of successful applications of the operations in Rule 1 – Rule 8 is \( O(n) \). Alber et al. [2] also showed that Rule 1 and Rule 2 can be executed in time \( O(n^2) \) when the graph is planar. Similarly, one can show that Rules 3–8 can also be executed in \( O(n^2) \) time (we leave the verification of this simple fact to the interested reader). This, together with the fact that the total number of successful applications of all the rules is \( O(n) \), implies that the time needed to construct \( G' \) is \( O(n^3) \).

To show parts (2) and (3) of the theorem, we prove that after the execution of any of the rules, the resulting graph satisfies conditions (2) and (3) in the theorem. The proof will then follow by an inductive argument on the number of applications of the rules. Denote by \( H \) the graph before a rule is executed, and by \( H' \) the resulting graph after the rule is executed. Denote by \( D \) a minimum dominating set for \( H \) excluding all white vertices in \( H \). Initially, \( H = G \) and all vertices in \( H \) are black. Thus, \( H \) trivially satisfies conditions (2) and (3) in the theorem. Suppose now that one of the rules is executed on a graph \( H \) satisfying conditions (2) and (3) in the theorem to yield the graph \( H' \). We need to show that \( H' \) satisfies conditions (2) and (3) as well.
Suppose that Rule 1 is executed. The same argument used in [2] shows that $\gamma(H) = \gamma(H')$.\(^1\) What is left is showing that $H'$ has a minimum dominating set consisting only of black vertices. Let $D$ be a minimum dominating set for $H$ consisting of black vertices. Since $N_3(v) \neq \emptyset$, $D$ must contain a vertex in $N_2(v) \cup N_3(v) \cup \{v\}$. If $D$ contains a vertex in $N_2(v) \cup N_3(v)$, then clearly this vertex can be replaced by $v$ which is black. Thus we can assume that $D$ contains $v$ and no vertex in $N_3(v) \cup N_2(v)$. Then $D$ is also a dominating set for $H'$ consisting only of black vertices, and since $\gamma(G) = \gamma(H) = \gamma(H')$, $D$ is a minimum dominating set for $H'$. It follows that $H'$ satisfies conditions (2) and (3). The proof of Rule 2 is of the same flavor.

If Rule 3 is executed, then the black vertices in the set $N_2(v) \cup N_3(v)$, where $v$ is black, will be colored white, and the edges between the white vertices are removed. It suffices to show that after the coloring of one vertex $x$ in $N_2(v) \cup N_3(v)$ white, and removing the edges between $x$ and the other white vertices, conditions (2) and (3) still hold (the same argument can then be applied repetitively to every such vertex). By our inductive hypothesis, before the application of Rule 3 to $H$, $H$ had a minimum dominating set $D$ of size equal to $\gamma(G)$ that excludes all white vertices in $H$. If $D$ contains $x$, we can substitute $x$ with $v$ and have a minimum dominating set of $H$ consisting only of black vertices in $H$. Thus, we can assume, without loss of generality, that $D$ does not include $x$. Since $x$ is the only vertex whose color has changed to white, $D$ consists only of black vertices in $H'$. Moreover, it is not difficult to see that $D$ is also a dominating set in $H'$ since the edges removed from $H$ are not used to dominate any vertices in $H$ (these edges were incident on vertices which are excluded from $D$). Since by removing edges from the graph the size of the minimum dominating set can only increase, we conclude that $D$ is a minimum dominating set for $H'$ excluding all white vertices, and $\gamma(H') = \gamma(H) = \gamma(G)$.

Suppose Rule 4 is executed. Similarly, we only need to show that conditions (2) and (3) still hold after one vertex $x$ is colored white. If $D$ contains $x$, then by the assumption in Rule 4, $D$ must also contain at least another vertex $x'$ in $N_2(v, w) \cup N_3(v, w) \cup \{v, w\}$ to dominate $N_3(v, w)$. This is true since only vertices in $N_2(v, w) \cup N_3(v, w) \cup \{v, w\}$ can dominate vertices of $N_3(v, w)$. In such case we can substitute $x$ and $x'$ by $v$ and $w$ and have a minimum dominating set that consists of only black vertices in $H$. Since $x$ is the only vertex whose color has changed to white, $D$ excludes all white vertices in $H'$. It is easy to see that the edges that connect white vertices in $H$ are not used by $D$ to dominate any vertex. By an argument similar to the above, it follows that $D$ is a minimum dominating set for $H'$ excluding all white vertices in $H'$, and $\gamma(H') = |D| = \gamma(H) = \gamma(G)$.

Suppose Rule 5 is executed and a vertex $x$ is colored white. Let $R = R(v, w)$ be the quasimultiple region that was being processed in the rule, and note that all the vertices in $R^+(v, w)$ are connected to $v$. Let the boundary of $R$ be $(v, y, w, z, v)$. Let $D$ be a dominating set for $H$ consisting of only black vertices. If $D$ contains $x$, then $D$ must contain another black vertex $x' \in R(v, w)$ in $N_2(v, w) \cup N_3(v, w) \cup \{v, w, y, z\}$ to dominate the vertices in $N_2(v, w) \cup N_3(v, w)$ that are strictly inside $R$. Observe that the only vertex that can be dominated by $x$ and not by $v$ is $w$. We distinguish the following cases:

**Case 1.** $x' = v$. Since at least one vertex $r \in \{y, w, z\}$ must be black and, all of them dominate $w$, we can substitute $x$ by $r$ (note that $x$ is dominated by $v$) to obtain a dominating set consisting of black vertices that excludes $x$.

**Case 2.** $x' \neq v$. If $x'$ does not dominate $w$, then $x'$ must be one of those vertices in $R$ that connect only to $v$ and to the vertices on the boundary of $R$ other than $w$. Since all such boundary vertices are dominated by $v$, and $x'$ is dominated by $v$ as well, we can substitute $x'$ by $v$ in $D$ and the case

\(^1\)This can be easily verified by the reader.
reduces to Case 1 above. If \( x' \) dominates \( w \), then we can substitute \( x \) by \( v \) to get a dominating set consisting of black vertices that excludes \( x \).

Thus, we can assume, without loss of generality, that \( D \) does not include \( x \). Since \( x \) is the only vertex whose color has changed to white, \( D \) consists only of black vertices in \( H' \). By an argument similar to above, it follows that \( D \) is a minimum dominating set for \( H' \) excluding all white vertices in \( H' \), and \( \gamma(H') = |D| = \gamma(H) = \gamma(G) \).

Suppose Rule 6 is executed and a vertex \( v \) is removed as described in the rule. Let \( D \) be a minimum dominating set for \( H \) excluding all white vertices in \( H \). Thus \( D \) does not contain \( v \). Since \( v \) is the only vertex removed, and no vertices are colored, it follows that \( D \) is a dominating set for \( H' \) excluding all white vertices in \( H' \). What is left is proving that \( D \) is a minimum dominating set for \( H' \). Suppose that \( H' \) has a minimum dominating set \( D' \) of size strictly smaller than \( D \). Then \( D' \) has to cover \( u \), and hence, \( D' \) either contains \( u \) or a neighbor of \( u \) in \( H' \). If \( D' \) contains \( u \), since every neighbor of \( u \) is also a neighbor of \( w \), and \( u \) is a neighbor of \( w \), \( (D' \cup \{w\}) - \{u\} \) is a minimum dominating set for \( H \) of size smaller than \( D \), a contradiction (note that since \( w \) is a neighbor of \( u \), \( w \) is a neighbor of \( v \) as well, and hence, dominates \( v \)). On the other hand, if \( D' \) contains a neighbor of \( u \), since \( N(u) \subseteq N(v) \), \( D' \) also contains a neighbor of \( v \) in \( H \), and hence dominates \( v \). Thus, \( D' \) is a dominating set for \( H \) of size smaller than \( D \), a contradiction. It follows that \( |D| = \gamma(H') = \gamma(H) = \gamma(G) \).

Suppose Rule 7 is executed on a black vertex \( v \), and all vertices in \( W(v) \) were removed as described in the rule. Let \( D \) be a minimum dominating set for \( H \) excluding all white vertices in \( H \). Thus, \( D \) does not contain any vertex in \( W(v) \). Since \( W(v) \) are the only vertices that were removed, and no vertices in the graph were colored, it follows that \( D \) is a dominating set for \( H' \) excluding all white vertices in \( H' \). What is left is proving that \( D \) is a minimum dominating set for \( H' \). Suppose that \( H' \) has a minimum dominating set \( D' \) of size strictly smaller than \( D \). Then \( D' \) has to cover \( v \). Hence \( D' \) either contains \( v \), or a neighbor of \( v \) in \( B(v) \) because all the vertices in \( W(v) \) were removed. In either case, \( D' \) dominates all the removed vertices in \( W(v) \) since every vertex in \( W(v) \) is adjacent to all vertices in \( B(v) \). Therefore \( D' \) is also a dominating set for \( H \) of size smaller than \( D \), a contradiction. It follows that \( |D| = \gamma(H') = \gamma(H) = \gamma(G) \).

To prove the statement for Rule 8, let \( D \) be a minimum dominating set for \( H \) excluding all white vertices in \( H \). Again, \( D \) is a dominating set for \( H' \) excluding all white vertices in \( H' \). Let \( D' \) be a minimum dominating set for \( H' \) and suppose, to get a contradiction, that \( |D'| < |D| \). Without loss of generality, we can assume that \( D' \) contains either \( v \) or \( w \) (or both), otherwise, \( D' \) has to contain both \( u \) and \( u' \), which can be substituted by \( v \) and \( w \). Now \( D' \) is also a dominating set for \( H \) of smaller size than \( D \), a contradiction. It follows that \( D \) is a minimum dominating set for \( H' \) excluding all white vertices in \( H' \), and \( \gamma(H') = \gamma(H) = \gamma(G) \).

To prove part (4) of the theorem, note the following: (1) from a minimum dominating set for \( G' \) one can construct in \( O(n) \) time a minimum dominating set for \( G' \) containing only black vertices (this can be achieved by associating, during the reduction phase, with the vertices colored white the black vertices that can replace them); and (2) a minimum dominating set for \( G' \) consisting only of black vertices is also a minimum dominating set for \( G \). This completes the proof.

3 A problem kernel

Let \( G \) be a reduced graph, and let \( D \) be a minimum dominating set for \( G \) consisting of black vertices such that \( |D| = k \). In this section, we will show that the number of vertices \( n \) in \( G \) is bounded by
67k. We start with the following propositions.

**Proposition 3.1** If there is no simple black vertex strictly inside a quasi-simple region $R = R(v, w)$, then $V(R)$ contains at most two simple white vertices.

**Proof.** Suppose, to get a contradiction, that $R$ has more than two simple white vertices that are strictly inside. Let $a$, $b$, and $c$ be three such vertices. Since all the three vertices are simple, one vertex must be engulfed within the area determined by $v$, $w$, and the other two simple white vertices. Suppose that $b$ is situated within the area $(v, a, w, c, v)$. Now all the vertices strictly inside $R$ must belong to $N_2(v, w) \cup N_3(v, w)$. Moreover, if a vertex $d \notin \{a, b, c\}$ is strictly inside $R$, then $d$ cannot dominate all of $\{a, b, c\}$, and hence, $d$ cannot dominate all the vertices in $N_2(u, v) \cup N_3(v, w)$ that are strictly inside $R$. Consequently, by Rule 5, all vertices strictly inside $R$ will be colored white (note that the color of both $v$ and $w$ must be black since there is a simple white vertex that is connected to both $v$ and $w$). Since no edges exist between white vertices, the degree of $b$ is exactly 2. Now $|W(v, w)| > 2$ because $\{a, b, c\} \subseteq W(v, w)$. But this makes Rule 8 applicable contradicting the fact that $G$ is reduced. This completes the proof. 

**Proposition 3.2** Let $R = R(v, w)$ be a quasi-simple region where the color of $v$ is black, then $V(R)$ has at most 4 simple vertices.

**Proof.** Suppose first that $R$ has six or more simple vertices. Let $S$ be the set of those simple vertices that are strictly inside $R$. Then $|S| \geq 4$. Since the vertices in $S$ are simple and hence connect to both $v$ and $w$, it is obvious that no vertex lying strictly inside $R$ can dominate all vertices in $S$. But $S$ is a subset of those vertices in $N_2(v, w) \cup N_3(v, w)$ that are strictly inside $R$, it follows that no vertex that is strictly inside $R$ can dominate all vertices in $N_2(v, w) \cup N_3(v, w)$. Now all vertices that lie strictly inside $R$ belong to $N_2(v, w) \cup N_3(v, w)$, thus, by Rule 5, all vertices strictly inside $R$ must be white. Noting that $|S| \geq 4$, and all vertices in $S$ are simple white vertices, this contradicts Proposition 3.1.

Suppose now that $R$ has five simple vertices. Let $a$, $b$, and $c$, be the three simple vertices that lie strictly inside $R$. By an argument similar to the above, we can assume that vertex $b$ is engulfed within the area determined by $v$, $a$, $w$, and $c$. Again all the vertices strictly inside $R$ must belong to $N_2(v, w) \cup N_3(v, w)$. Since $a$ does not dominate $c$, and vice versa, it follows that $a$ and $c$ are colored white by Rule 5. Now $a$, $b$, and $c$ are the only simple vertices strictly inside $R$, by Proposition 3.1, no three simple white vertices can be contained in $R$. This forces $b$ to be black, and to be connected to both $a$ and $c$ (otherwise $b$ would be colored white by Rule 5). Now all other non-simple vertices in $R$ must be connected to the boundary, and hence cannot be connected to $b$ (all the vertices other than $a$ and $c$ which can be connected to $b$ have to belong to the area engulfed by $(v, a, w, c)$ and cannot be connected to the boundary). Thus, $W(b) = \{a, c\}$, and every vertex in $W(b)$ is connected to all vertices in $B(b) = \{v, w\}$ (note that since $a$ and $c$ are white, and are connected to $w$, $w$ must be black). By Rule 7, $W(b) = \{a, c\}$ should have been removed at this point, a contradiction. Thus, $R$ has at most four simple vertices and the proof is complete. 

The following definitions are adopted from [2].

Given any dominating set $D$ in a graph $G$, a $D$-region decomposition of $G$ is a set $\mathcal{R}$ of regions between pairs of vertices in $D$ such that:

1. For any region $R = R(v, w)$ in $\mathcal{R}$, no vertex in $D$ is in $V(R)$. That is, a vertex in $D$ can only be an endpoint of a region in $\mathcal{R}$. 

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2. No two distinct regions $R_1, R_2 \in \mathbb{R}$ intersect. However, they may touch each other by having common boundaries.

Note that all the endpoints of the regions in a $D$-region decomposition are vertices in $D$. For a $D$-region decomposition $\mathbb{R}$, define $V[\mathbb{R}] = \bigcup_{R \in \mathbb{R}} V[R]$. A $D$-region decomposition is maximal, if there is no region $R$ such that $\mathbb{R}' = \mathbb{R} \cup R$ is a $D$-region decomposition with $V[\mathbb{R}] \subseteq V[\mathbb{R}']$.

For a $D$-region decomposition $\mathbb{R}$, associate a planar graph $G_{\mathbb{R}}(V_{\mathbb{R}}, E_{\mathbb{R}})$ with possible multiple edges, where $V_{\mathbb{R}} = D$, and such that there is an edge between two vertices $v$ and $w$ in $G_{\mathbb{R}}$ if and only if $R(v, w)$ is a region in $\mathbb{R}$. A planar graph with multiple edges is called thin, if there is a planar embedding of the graph such that for any two edges $e_1$ and $e_2$ between two distinct vertices $v$ and $w$ in the graph, there must exist two more vertices which sit inside the disjoint areas of the plane enclosed by $e_1$ and $e_2$.

Alber et al. [2] showed that the number of edges in a thin graph of $n$ vertices is bounded by $3n - 6$. They also showed that for any plane graph $G$ and a dominating set $D$ of $G$, there exists a maximal $D$-region decomposition for $G$ such that $G_{\mathbb{R}}$ is thin. Since the maximal $D$-region decomposition in [2] starts with any dominating set $D$ and is not affected by the color a vertex can have, the same results in [2] hold true for our reduced graph $G$ whose vertices are colored black/white, and with a minimum dominating set $D$ consisting only of black vertex. The above discussion is summarized in the following proposition.

**Proposition 3.3** Let $G$ be a reduced graph and $D$ a dominating set of $G$ consisting of black vertices. Then there exists a maximal $D$-region decomposition $\mathbb{R}$ of $G$ such that $G_{\mathbb{R}}$ is thin.

**Corollary 3.4** Let $G$ be a reduced graph with a minimum dominating set $D$ consisting of $k$ black vertices, and let $\mathbb{R}$ be a maximal $D$-region decomposition of $G$ such that $G_{\mathbb{R}}$ is thin. Then the number of regions in $\mathbb{R}$ is bounded by $3k - 6$.

**Proof.** The number of regions in $\mathbb{R}$ is the number of edges in $G_{\mathbb{R}}$. Since $G_{\mathbb{R}}$ has $|D| = k$ vertices, by [2], the number of edges in $G_{\mathbb{R}}$ is bounded by $3k - 6$. \qed

In the remainder of this section, $\mathbb{R}$ will denote a maximal $D$-region decomposition of $G$ such that $G_{\mathbb{R}}$ is thin. Let $u$ and $v$ be two vertices in $G$. We say that $u$ and $v$ are boundary-adjacent if $(u, v)$ is an edge on the boundary of some region $R \in \mathbb{R}$. For a vertex $v \in G$, denote by $N^*(v)$ the set of vertices that are boundary-adjacent to $v$. Note that for a vertex $v \in D$, since $v$ is black, by Rule 3, all vertices in $N_2(v) \cup N_3(v)$ must be white.

**Proposition 3.5** Let $v \in D$. The following are true.

(a) (Lemma 6, [2]) Every vertex $u \in N_1(v)$ is in $V[\mathbb{R}]$.

(b) The vertex $v$ is an endpoint of a region $R \in \mathbb{R}$. That is, there exists a region $R = R(x, y) \in \mathbb{R}$ such that $v = x$ or $v = y$.

(c) Every vertex $u \in N_2(v)$ which is not in $V[\mathbb{R}]$ is connected only to $v$ and to vertices in $N^*(v)$.

**Proof.** To prove part (a), let $u \in N_1(v)$, and assume to get a contradiction, that $u \notin V[\mathbb{R}]$. By the definition of $N_1(v)$, there exists a vertex $u' \in N(u)$ such that $u' \notin N[v]$. Since $u'$ needs to be dominated by $D$, either $u' \in D$ or $u'$ is dominated by a vertex $w \in D$ with $w \neq v$. If $u' \in D$, consider the degenerated region consisting of the path $(v, u, u')$. Since $\mathbb{R}$ is maximal and $u \notin V[\mathbb{R}]$,
this degenerated region must cross a region $R \in \mathcal{R}$. But this would imply that $u \in V[R]$, and hence $u \in V[\mathcal{R}]$, a contradiction. Assume now that $u'$ is dominated by a vertex $w \in D$ different from $v$. Consider the degenerated region consisting of the path $(v, u, u', w)$. By maximality of $\mathcal{R}$, this path must cross a region $R = R(x, y) \in \mathcal{R}$, where $x, y \in D$. Since we assumed that $u \notin V[\mathcal{R}]$, the edge $(u', w)$ crosses $R$ which implies that $w$ lies in $V[R]$. According to the definition of a $D$-region decomposition, the only vertices in $D$ that are in $V[R]$ are $x$ and $y$, hence, $w \in \{x, y\}$. Without loss of generality, assume that $w = x$. At the same time, $u'$ must lie on the boundary of $R$, otherwise $u \in V[R]$. By the definition of a region, there exists a path $P$ between $w$ and $y$ of length at most three that goes through $u'$ and that is part of the boundary of $R$. Thus, $u'$ must be a neighbor of $y$, otherwise, the edge $(u', w)$ would not cross $R$, and would lie on the boundary of $R$. But then the degenerated region $R'$ formed by the path $(y, u', u, v)$ is a region between two vertices $y$ and $v$ in $D$ which does not cross any region in $\mathcal{R}$ (it only touches $R$). This contradicts the maximality of $\mathcal{R}$ since $\mathcal{R}' = \mathcal{R} \cup \{R'\}$ is a $D$-region decomposition containing $u \notin \mathcal{R}$, and hence, is strictly larger than $\mathcal{R}$.

To prove (b), suppose to get a contradiction that $v$ is not the endpoint of any region in $\mathcal{R}$. Since $v \in D$, and by the definition of a region, $v$ must be outside every region in $\mathcal{R}$. Now $v$ must have a vertex in $N_1(v)$, otherwise, all vertices in $N(v)$ would be white, and hence removed by Rule 7 (we assume, without loss of generality, that $G$ does not contain any isolated vertices). Let $u \in N_1(v)$. By part (a) above, $u$ must belong to some region $R = R(x, y)$. Observe that $u$ must be on the boundary of $R$, otherwise $v$ would be a vertex in $V[R]$. Again, by the definition of a region, $u$ is either boundary-adjacent to $x$ or to $y$. Suppose, without loss of generality, that $u$ is boundary-adjacent to $x$. But then the degenerated region formed by $(x, u, v)$ does not cross $\mathcal{R}$ (it only touches $R$), contradicting the maximality of $\mathcal{R}$.

To prove part (c), let $u$ be a vertex in $N_2(v)$, and suppose $u$ is connected to a vertex $w \neq v$ such that $w \notin N^*(v)$. Note that $w$ must be in $N_1(v)$, and hence, by part (a) above, must belong to some region $R = R(x, y)$. Since $u \notin \mathcal{R}$, $w$ cannot be inside $R$, and hence, is on the boundary of $R$. Moreover, by the definition of a region, $w$ must be boundary-adjacent to either $x$ or $y$. Without loss of generality, assume $w$ is boundary adjacent to $x$. Now $w \notin N^*(v)$, so $w$ cannot be boundary-adjacent to $v$, and $x \neq v$. Consider the degenerated region formed by $(v, u, w, x)$. This region cannot cross any region in $\mathcal{R}$, otherwise it crosses it via $(u, w)$ and $u$ would be in $V[\mathcal{R}]$. But this contradicts the maximality of $\mathcal{R}$ since $u \notin V[\mathcal{R}]$.

Let $x$ be a vertex in $G$ such that $x \notin V[\mathcal{R}]$. Then by part (b) in Proposition 3.5, $x \notin D$. Thus, $x \in N(v)$ for some black vertex $v \in D \subseteq V[\mathcal{R}]$. By part (a) in Proposition 3.5, $x \notin N_1(v)$, and hence, $x \in N_2(v) \cup N_3(v)$. By Rule 3, the color of $x$ must be white. Let $R = R(v, w)$ be a region in $V[\mathcal{R}]$ of which $v$ is an endpoint (such a region must exist by part (b) of Proposition 3.5). We distinguish two cases.

**Case A.** $x \in N_3(v)$. Since $v$ is black, by Rule 1, this is only possible if $\text{deg}(x) = 1$ and $N_2(v) = \emptyset$ (in this case $x$ will be the white vertex added by the rule). In such case it can be easily seen that we can flip $x$ and place it inside $R$ without affecting the planarity of the graph.

**Case B.** $x \in N_2(v)$. Note that in this case $N_3(v) = \emptyset$, and $x$ is only connected to $v$ and $N^*(v)$ by part (c) in Proposition 3.5. If $\text{deg}(x) = 2$, by a similar argument to **Case A** above, $x$ can be flipped and placed inside $R$. 


According to the above discussion, it follows that the vertices in \( G \) can be classified into two categories: (1) those vertices that are in \( V[\mathbb{R}] \); and (2) those that are not in \( V[\mathbb{R}] \), which are those vertices of degree larger than two that belong to \( N_2(v) \) for some vertex \( v \in D \), and in this case must be connected only to vertices in \( N^*(v) \). To bound the number of vertices in \( G \) we need to bound the number of vertices in the two categories. We start with the vertices in category (2).

Let \( O \) denote the set of vertices in category (2). Note that all vertices in \( O \) are white, and no two vertices \( u \) and \( v \) in \( O \) are such that \( N(u) \subseteq N(v) \). To see why the latter statement is true, note that every vertex in \( O \) must be in \( N_2(w) \) for some black vertex \( w \in D \). So if \( N(u) \subseteq N(v) \), then by Rule 6, \( v \) would have been removed from the graph. To bound the number of vertices in \( O \), we will bound the number of vertices in \( O \) that are in \( N_2(v) \) where \( v \in D \). Let us denote this set by \( N^1(v) \). Let \( N^*_i(v) \) be the set of vertices in \( N^*(v) \) that are neighbors of vertices in \( N^1(v) \). Note that every vertex in \( N^1(v) \) has degree \( \geq 3 \), is connected only to \( v \) and to \( N^*_i(v) \), and no two vertices \( x \) and \( y \) in \( N^1(v) \) are such that \( N(x) \subseteq N(y) \).

**Assumption 3.6** We can assume that: (1) every vertex in \( N^1(v) \) has degree exactly 3; (2) no two vertices \( x \) and \( y \) in \( N^1(v) \) are such that \( N(x) \subseteq N(y) \); and (3) vertices in \( N^1(v) \) are only connected to \( v \) and \( N^*_i(v) \).

**Proof.** Since properties (2) and (3) are already satisfied by the vertices in \( N^1(v) \), we only need to show how we can make the vertices in \( N^1(v) \) satisfy property (1) without reducing their number, and without affecting properties (2) and (3). To satisfy property (1), we will remove some edges between vertices in \( N^1(v) \) and \( N^*_i(v) \) without affecting the other properties. This can be done as follows. List the vertices in \( N^1(v) \) in an arbitrary order \( (u_1, \ldots, u_p) \). Start by picking \( u_1 \), then choose any two neighbors of \( u_1 \) in \( N^*_1(v) \) and remove all edges that join \( u_1 \) to all its neighbors other than \( v \) and these two chosen neighbors. Inductively, suppose we have processed vertex \( u_{i-1} \), we process vertex \( u_i \) as follows. Pick two neighbors \( w_i^1 \) and \( w_i^2 \) of \( u_i \) in \( N^*_i(v) \) such that no vertex in \( \{u_1, \ldots, u_{i-1}\} \) has both \( w_i^1 \) and \( w_i^2 \) as its picked neighbors. Delete all the edges that join \( u_i \) to all vertices other than \( v \), \( w_i^1 \), and \( w_i^2 \). We need to show that it is always possible to carry out this step. Suppose not, and let \( i \) be the smallest index such that this is not possible. It is easy to verify using the facts that every vertex in \( N^1(v) \) has degree larger than two, and no two vertices \( x \) and \( y \) are such that \( N(x) \subseteq N(y) \), that \( i > 3 \). Note that \( deg(u_i) > 3 \), otherwise, since this step cannot be carried out successfully, there must exist three distinct vertices \( u_p, u_q, u_s \in \{u_1, \ldots, u_{i-1}\} \) such that \( \{a, b\} \subseteq N(u_p) \), \( \{a, c\} \subseteq N(u_q) \), and \( \{b, c\} \subseteq N(u_s) \). Consider the subgraph \( H \) of \( G \) induced by the set of vertices \( \{v, u_p, u_q, u_s, u_i, a, b, c\} \). Then the following is true about the vertices in \( H \): (1) \( u_i, u_p, u_q, u_s, a, b, c \) are neighbors of \( v \) in \( H \); (2) \( v, a, b, c \) are neighbors of \( u_i \) in \( H \); (3) \( v, a, b \) are neighbors of \( u_p \) in \( H \); (4) \( v, a, c \) are neighbors of \( u_q \) in \( H \); (5) \( v, b, c \) are neighbors of \( u_s \) in \( H \); (6) \( v, u_i, u_p, u_q \) are neighbors of \( a \) in \( H \); (7) \( v, u_i, u_p, u_s \) are neighbors of \( b \) in \( H \); and (8) \( v, u_i, u_q, u_s \) are neighbors of \( c \) in \( H \). Using all this information, it is not difficult to verify that \( H \) is non-planar (identify vertex \( a \) with vertex \( b \) along the path \( (a, u_p, b) \) to obtain a copy of \( K_{3,3} \)), contradicting the planarity of \( G \). This completes the proof.

**Proposition 3.7** \( |N^1(v)| \leq 3/2|N^*_i(v)| \).

**Proof.** To simplify the counting, by Assumption 3.6, we can assume that: every vertex in \( N^1(v) \)
has degree exactly 3; no two vertices $x$ and $y$ in $N^\uparrow(v)$ are such that $N(x) \subseteq N(y)$; and vertices in $N^\uparrow(v)$ are only connected to $v$ and $N_2^\uparrow(v)$. Let $x$ be the number of vertices in $N_1^\uparrow(v)$, and let $f(x) = |N^\uparrow(v)|$. We will show that $f(x) \leq 3/2(x - 1)$. We proceed by induction on $x$. If $x = 1$, it is clear that $f(x) = 0 \leq 3/2(x - 1)$ since by Assumption 3.6, each vertex in $N^\uparrow(v)$ has degree exactly 3. If $x = 2$, then clearly $f(x) \leq 1 \leq 3/2(x - 1)$ since at most one vertex can be connected to $v$ and the two vertices in $N_1^\uparrow(v)$. If $x = 3$, then $f(x) \leq 3$ since at most three vertices can be connected to $N_1^\uparrow(v)$ without violating properties (1) – (3) above, each connected to $v$ and to two other vertices in $N_1^\uparrow(v)$. Inductively, suppose that if $N_1^\uparrow(v)$ contains $y$ vertices with $3 \leq y < x$, then the number of vertices $f(y)$ in $N^\uparrow(v)$ satisfies $f(y) \leq 3/2(y - 1)$. Suppose now that $|N_1^\uparrow(v)| = x$.

Let $u$ be a vertex in $N^\uparrow(v)$, and let $a, b$ be its neighbors in $N_1^\uparrow(v)$. The vertex $u$ is called hollow if the interior of the region enclosed by $(u, a, b, u)$ contains no vertices of $N_1^\uparrow(v)$. If every vertex in $N^\uparrow(v)$ is hollow, then it is clear that $f(x) \leq x \leq 3/2(x - 1)$ for $x > 3$, and the bound $f(x) = x$ is attained when there are $x$ vertices in $N^\uparrow(v)$, and every vertex $u$ in $N^\uparrow(v)$ is adjacent to $v$ and the two neighbors $a$ and $b$ in $N_1^\uparrow(v)$ immediately to the left and right in the clockwise (or anticlockwise) ordering, respectively, of $u$ in the embedding. Suppose now that there is a vertex $u \in N^\uparrow(v)$ such that $u$ is not hollow. The edges $(u, a), (u, v), (u, b), (v, a), (v, b)$ separate the plane into three faces: $F_1$ enclosed by the cycle $(u, a, v, u), F_2$ enclosed by the cycle $(u, v, b, u), F_3$ is the outer face determined by the cycle $(u, a, v, b, u)$. Let $x_1$ be the number of vertices in $N_1^\uparrow(v)$ that are in $F_1$ including the boundary, $x_2$ that in $F_2$, and $x_3$ that in $F_3$. Note that $1 \leq x_1 < x$ since $a \in F_1$ and $b \notin F_1$, $1 \leq x_2 < x$ since $b \in F_2$ and $a \notin F_2$, and $2 \leq x_3 < x$ since $a$ and $b$ are in $F_3$ and at least one vertex in $N_1^\uparrow(v)$ is not in $F_3$ since $u$ is hollow, and hence, the interior of the face $(u, a, v, b, u)$ contains at least one vertex in $N_1^\uparrow(v)$. Moreover, $x_1 + x_2 + x_3 = x + 2$, since $a$ and $b$ are the only vertices counted twice when we add the vertices in $N_1^\uparrow(v)$ that are in $F_1, F_2,$ and $F_3$.

Now every vertex in $N^\uparrow(v)$ is either: (1) connected to two vertices in $N_1^\uparrow(v)$ in $F_1$, (2) connected to two vertices in $N_1^\uparrow(v)$ in $F_2$, or (3) connected to two vertices in $N_1^\uparrow(v)$ in $F_3$. Note that vertex $u$ satisfies property (3). Since $x_1, x_2, x_3 < x$, by the inductive hypothesis, the number of vertices satisfying (1) is bounded by $f(x_1) \leq 3/2(x_1 - 1)$, the number of vertices satisfying (2) is bounded by $f(x_2) \leq 3/2(x_2 - 1)$, and the number of vertices satisfying (3) is bounded by $f(x_3) \leq 3/2(x_3 - 1)$. Now $f(x) \leq f(x_1) + f(x_2) + f(x_3) \leq 3/2(x_1 + x_2 + x_3) - 9/2 = 3x/2 - 3/2 = 3/2(x - 1)$. This completes the proof.

Lemma 3.8

The number of vertices in category (2) (i.e., the number of vertices not in $V[\mathcal{R}]$) is bounded by $18k$.

**Proof.** Let $v$ and $w$ be any two distinct vertices in $D$ and observe the following. First, $N^\uparrow(v) \cap N^\uparrow(w) = \emptyset$, because if $u \in N^\uparrow(v) \cap N^\uparrow(w)$ then $(v, u, w)$ would be a degenerated region with $u \notin V[\mathcal{R}]$ contradicting the maximality of $\mathcal{R}$. Second, from the first observation it follows that $w \notin N_1^\uparrow(v)$ and $v \notin N_1^\uparrow(w)$ (in general no vertex $a \in D$ belongs to $N_1^\uparrow$ for any vertex $b \in D$); otherwise, there exists a vertex $u \in N^\uparrow(v)$ that is connected to $w$, and hence $u \cap N^\uparrow(v) \cap N^\uparrow(w)$, contradicting the first observation. Third, $N_1^\uparrow(v) \cap N_1^\uparrow(w) = \emptyset$; otherwise, there exists a vertex $u \in N_1^\uparrow(v) \cap N_1^\uparrow(w)$ that is connected to a category-(2) vertex $a \in N^\uparrow(v)$ (or $b \in N^\uparrow(w)$) and the degenerated region $(v, a, u, w)$ (or $(w, b, u, v)$) would contain the vertex $a \notin \mathcal{R}$ (or $b \notin \mathcal{R}$), contradicting the maximality of $\mathcal{R}$.

Let $B$ be the number of vertices not in $D$ that are boundary-adjacent to vertices in $D$ (i.e., in $N^\uparrow(v) - D$ for some $v \in D$). Combining the above observations with Proposition 3.7, it follows that the number of category-(2) vertices is
\[ \sum_{v \in D} |N^*(v)| \leq \frac{3}{2} \sum_{v \in D} |N^*(v)| \leq 3B/2 \]

According to the definition of a region, each region in \( \mathcal{R} \) has at most six vertices on its boundary two of which are vertices in \( D \). Thus, each region in \( \mathcal{R} \) can contribute with at most four vertices to \( B \). By Corollary 3.4, the number of regions in \( \mathcal{R} \) is bounded by \( 3k-6 \). It follows that \( B \leq 12k-24 \), and hence, the number of category-(2) vertices is bounded by \( 18k-36 < 18k \). This completes the proof.

To bound the number of vertices in category (1), fix a region \( R(v, w) \) between \( v, w \in D \). Without loss of generality, assume the boundary of \( R \) is determined by the two paths \( \langle v, v_1, w_1, w \rangle \) and \( \langle v, v_2, w_2, w \rangle \). Note that all vertices in \( V(R) \) belong to \( N(v, w) \), and that \( v_1, v_2 \in N^*(v) \), and \( w_1, w_2 \in N^*(w) \). If there is a degree-1 vertex \( x \) connected to \( v \) (resp. \( w \) ), then this vertex is in \( N_3(v) \) (resp. \( N_3(w) \)) and must be colored white by Rule 3. Similarly, if there exists a degree-2 vertex \( y \) that is connected to \( v \) and either \( v_1 \) or \( v_2 \) (resp. \( w \) and either \( w_1 \) or \( w_2 \)), then \( y \) is in \( N_2(v) \) (resp. \( N_2(w) \)) and must be colored white by Rule 3. Now if a degree-1 white vertex is connected to \( v \), then by Rule 6, \( N^*(v) = \emptyset \). During the process of counting the number of vertices in \( N^*(v) \), we bounded the number of vertices in \( N^*(v) \) by \( 3|N^*(v)|/2 \). This can be looked at as each vertex in \( N^*(v) \) contributing \( 3/2 \) vertices to \( |N^*(v)| \). So if a degree-1 white vertex is connected to \( v \) (note that at most one degree-1 vertex can be connected to \( v \) ), this means that \( N^*(v) \) which contains at least two vertices, will not contribute to the number of vertices in \( N^*(v) \), and hence, the bound on \( |N^*(v)| \) will be decreased by at least three. Similarly, if a degree-2 white vertex is connected to \( v \) and \( v_1 \) (or \( v_2 \) again, note that there can be at most one degree-2 vertex connected to \( v \) and to \( v_1 \) (or \( v_2 \)), then no vertex in \( N^*(v) \) can be connected to \( v_1 \) (or \( v_2 \)). This can be regarded as a reduction to the bound on \( |N^*(v)| \) by \( 3/2 \). Thus, if we use the upper bound on the number of vertices in category (2) computed above, we may assume without loss of generality that no degree-1 vertex is connected to \( v \) or \( w \), and that no degree-2 vertex is connected to \( v \) and \( v_1, v_2, w \), and \( w_1, w, w_2 \). We will also assume that the boundary of a region \( R(v, w) \) consists of exactly six distinct vertices, that is, the region is not a degenerate region. The case of a degenerate region yields a better bound on the number of vertices in the region. Let us call a region with all the above properties nice. We start with the following propositions.

**Proposition 3.9** Let \( (v, y, w, z, v) \) be the boundary of a quasi-simple region \( R = R(v, w) \), and suppose that \( v \) and \( y \) are black. Then there can be at most one white vertex in \( R^+ = R^+(v, w) \) that is connected to both \( v \) and \( y \).

**Proof.** Suppose, to get a contradiction, that there are at least two white vertices in \( R^+ \) that are connected to both \( v \) and \( y \). Since all the vertices in \( R^+ \) are white, and hence cannot be connected together, there must exist two white vertices \( a \) and \( b \) in \( R^+ \) satisfying that the area engulfed by \( (v, a, y, v) \) is empty, and the area engulfed by \( (v, b, y, v) \) contains only the vertex \( a \). Clearly, the degree of \( a \) is exactly 2, and \( a \) belongs to \( N_2(v) \cup N_3(v) \). Now since both \( a \) and \( b \) are connected to both \( v \) and \( y \), we have \( N(a) \subseteq N(b) \). Given the fact that \( v \) is black, this is a contradiction to Rule 6.

**Proposition 3.10** Let \( R = R(v, w) \) be a quasi-simple region, and suppose that \( v \) is black. Let \( (v, y, w, v) \) be the boundary of \( R \). If

(a) there are no simple vertices strictly inside \( R \), or
(b) there are simple vertices strictly inside $R$ and all vertices in $R^+ = R^+(v,w)$ are connected to $y$,

then $V(R) \cup \{w\}$ contains at most three white vertices. Moreover, if there are three white vertices in $V(R) \cup \{w\}$, then either $R^+ \neq \emptyset$, or there is a simple black vertex interior to $R$.

**Proof.** To prove that part (a) implies the statement of the proposition, suppose that there are no simple vertices lying strictly inside $R$. Then clearly all the white vertices in $V(R)$ come from $R^+ \cup \{y,z\}$. If $y$ is white, then no vertex in $R^+$ can be connected to $y$ because the vertices in $R^+$ are all white. On the other hand, since $R^+ \subseteq N_2(v) \cup N_2(v)$, if $y$ is black, by Proposition 3.9, at most one white vertex in $R^+$ can be connected to $y$. Similarly for $z$. Since every vertex in $R^+$ has to be connected to either $y$ or $z$ by the definition of a quasi-simple region, it follows from the above that $V(R)$ contains at most two white vertices, and hence $V(R) \cup \{w\}$ contains at most three white vertices. Now when $V(R) \cup \{w\}$ contains three white vertices, $w$ must be black, and hence, $y$ and $z$ are black. Thus, the two white vertices other than $w$ in $V(R) \cup \{w\}$ come from $R^+$, and $R^+ \neq \emptyset$.

To prove part (b), note first that, by Proposition 3.2, the number of simple vertices in $R$ including $y$ and $z$ is bounded by four. We will assume that the number of simple vertices in $R$ is exactly four. The cases when there are less than four simple vertices in $R$ are simpler, and yield the desired bound. Let $a$ and $b$ be the other two simple vertices, and assume that the four simple vertices $y, a, b, z$ appear in the preceding sequence in a clockwise (or anticlockwise) order around $v$. Observe that the white vertices in $V(R)$ come from $R^+ \cup \{y,a,b,z\}$. Also observe that since all the vertices in $R^+$ are white by the hypothesis of part (b), either $y$ is white and $R^+$ is empty, or $y$ is black and by Proposition 3.9, $R^+$ contains at most one vertex. It follows that the number of white vertices in $R^+ \cup \{y\}$ is bounded by one. Now suppose to get a contradiction that $V(R) \cup \{w\}$ contains four white vertices. Since no two white vertices are connected, and since all vertices in $\{a,b,z\}$ are connected to $w$, $w$ must be black and all the three vertices $a, b$, and $z$ must be white. But then the degree of $b$ is exactly 2, and $|W(v,w)| > 2$, contradicting Rule 8. To complete the proof, suppose that $V(R) \cup \{w\}$ contains exactly three white vertices, we need to show that either $R^+ \neq \emptyset$ or there exists a simple black vertex inside $R$. Suppose to get a contradiction that $R^+ = \emptyset$ and the interior vertices to $R, a, b, z$ are all white. Then $w$ must be black in this case and either $y$ or $z$ is white. Without loss of generality, assume $y$ is white. Since there are no edges between white vertices, the degree of $a$ must be two and $\{y,a,b\} \subseteq W(v,w)$, again a contradiction to Rule 8. \hfill $\square$

**Lemma 3.11** Let $R = R(v,w)$ be a nice region in $V[\mathcal{R}]$. The number of vertices in $V(R)$ is bounded by 16.

**Proof.** Every vertex in $V(R)$ is in $N(v,w)$, and hence, is either connected to $v$ or $w$. We distinguish two cases.

**Case 1.** $N_3(v,w) = \emptyset$. In this case every vertex in $V(R) - \{v_1,v_2,w_1,w_2\}$ has to be connected to at least one vertex in $\{v_1,v_2,w_1,w_2\}$ because $v_1, v_2, w_1, w_2$ are the only vertices in $V(R)$ that possibly belong to $N_1(v,w)$. Since $R$ is nice, the vertices in $V(R) - \{v_1,v_2,w_1,w_2\}$ can be classified into the following categories, where a vertex is assigned to the first category that it satisfies:

(i) vertices connected to $v$ and $v_1$, but not connected to $w_1$;
(ii) vertices connected to $v$ and $w_1$;
(iii) vertices connected to $v$ and $w_2$;

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(iv) vertices connected to \(v\) and \(v_2\);
(v) vertices connected to \(w\) and \(w_2\), but not connected to \(v_2\);
(vi) vertices connected to \(w\) and \(v_2\);
(vii) vertices connected to \(w\) and \(v_1\); and
(viii) vertices connected to \(w\) and \(w_1\).

Note that one of categories (ii) and (vii) must be empty, otherwise, according to our placement of
the vertices in the categories, we have two distinct vertices in \(V(R)\) other than \(v_1\) and \(w_1\), one of
them is connected to \(v\) and \(w_1\) and the other to \(w\) and \(v_1\), contradicting the planarity of the graph.
Similarly, one of categories (iii) and (vi) must be empty. Without loss of generality, assume that
categories (iii) and (vii) are empty. If, in addition, any of categories (ii) or (vi) is empty, then
the situation becomes simpler leading to a better bound on the number of vertices in \(V(R)\). Thus,
we will assume that both categories (ii) and (vi) are nonempty. Note also that since \(R\) is nice,
every vertex in category (i) must be connected to some vertices interior to \(R\). Since category (ii)
is nonempty, and by the planarity of \(G\), a vertex in category (i) is connected only to vertices in
category (ii). Moreover, since the vertices in category (i) are only connected to \(v\), and to neighbors
of \(v\) including \(v_1\), all these vertices belong to \(N_2(v)\). Since \(v\) is black, by Rule 3, all vertices
in category (i) must be white. Now the vertices in category (i) and category (ii), plus \(v\), \(v_1\), and
\(w_1\), form a quasi-simple region between \(v\) and \(w_1\). Moreover, all the vertices in
\(V(Q_1)\) except those in category (i) are simple vertices because all vertices in \(V(Q_1)\), except those
in category (i), have to be connected to both \(v\) and \(w_1\). Since \(v\) is black, by Proposition 3.2, the
number of vertices in \(V(Q_1)\) except those vertices in category (i), is bounded by 4. Now we bound
the number of vertices in category (i). Every vertex in category (i) is white and is connected to
\(v\) and \(v_1\). If category (i) is nonempty, then \(v_1\) must be black, and the vertices in category (i)
are white vertices in \(Q_1(v, w_1)\) that are connected to \(v\) and \(v_1\). It follows from Proposition 3.9 that
the number of vertices in category (i) is bounded by 1. This shows that the number of vertices
in \(V(Q_1)\) is bounded by five. By symmetry, the number of vertices in \(V(Q_2)\), where \(Q_2\) is the
quasi-simple region between \(w\) and \(v_2\) consisting of the vertices in category (v) and category (vi),
plus the vertices \(w, w_2, v_2\), is bounded by five. Now we bound the number of vertices in categories
(iv) and (viii). We have the following claim.

**Claim.** The number of vertices in category (iv) (resp. category (viii)) is bounded by 2. Moreover,
at most one vertex in category (iv) (resp. category (viii)) is white.

Consider the vertices in category (iv). Suppose that there are three or more vertices in category
(iv), and let \(a_1, a_2, a_3\) be three vertices in category (iv) such that: no vertex is engulfed in the area
of the embedding determined by \((v, a_1, v_2)\), \(a_1\) is the only vertex engulfed in the area determined
by \((v, a_2, v_2)\), and \(a_1\) and \(a_2\) are the only two vertices engulfed in the area determined by \((v, a_3, v_2)\).
Now \(a_1\) and \(a_2\) must belong to \(N_2(v)\). Since \(v\) is black, it follows from Rule 3 that \(a_1\) and
\(a_2\) must be white, and no edge exists between \(a_1\) and \(a_2\). But this means that \(N(a_1) \subseteq N(a_2)\),
and since \(a_1 \in N_2(v)\) and \(v\) is black, then according to Rule 6, this leads to a contradiction. It
follows that at most two vertices can be in category (iv). By symmetry, at most two vertices can
be in category (viii). Note also that it follows from the above proof that if there are exactly two
vertices \(a_1\) and \(a_2\) in category (iv) (resp. category (viii)), then at most one vertex in \(\{a_1, a_2\}\) can
be white. This proves the claim.

Now the vertices in \(V(R)\) consist of vertices of \(V(Q_1)\), vertices of \(V(Q_2)\), category (iv) and cat-
that the number of white vertices in the set $X$ is bounded by three. Moreover, the statement of the claim in category (iii) contains at most one white vertex. It follows that the number of vertices in $R$ is bounded by 16. See Figure 1 for an illustration.

**Case 2.** $N_3(v, w) \neq \emptyset$. Let $X$ be the set of white vertices in $N_2(v, w)$ that are in $V(R)$, $Y$ the set of black vertices in $N_2(v, w) \cup N_3(v, w)$ that are in $V(R)$, and $Z$ the set of white vertices in $N_3(v, w)$ that are in $V(R)$. We first draw few observations.

**Observation 1.** $|X| \leq 7$. We first show that $|X| \leq 8$. Remove the vertices in $N_3(v, w)$ interior to $R$, then define categories (i) -- (viii) as above. Similar to **Case 1**, we can assume that the vertices in category (i) and (ii), plus the vertices $v, v_1,$ and $w_1$, form a quasi-simple region $Q_1 = R(v, w_1)$, and those in category (v) and (vi), plus the vertices $w, w_2,$ and $v_2$, form a quasi-simple region $Q_2 = R(v, w_2)$. Since $X \subseteq N_2(v, w)$, every vertex in $X$ must belong to one of categories (i) -- (viii), or possibly to the set $\{v_1, v_2, w_1, w_2\}$. From the definition of categories (i) and (ii), vertices in category (i) form the set $Q_1^+ = Q_1^+(v, w_1)$ in the quasi-simple region $Q_1$, and all the vertices in $Q_1^+$ are connected to $v_1$. Now add the vertices in $N_3(v, w)$ back, and note that no black vertex in $N_3(v, w)$ that is not connected to $w_1$ resides in $Q_1$. The reason being that such a vertex would be in $N_2(v) \cup N_3(v)$ (otherwise, this vertex will have to be connected to $w_1$ — the only vertex in $Q_1$ possibly not in $N(v)$) and hence colored white by **Rule 3**. Now $Q_1$ plus the set of black vertices in $N_3(v, w)$ that reside in $Q_1$, minus the set of white vertices in $N_3(v, w)$ that reside in $Q_1$, satisfies condition (b) in Proposition 3.10, and the number of white vertices in $Q_1$ is bounded by three. Since no two white vertices are connected together, and hence the presence of the white vertices from $N_3(v, w)$ in $Q_1$ cannot increase the number of possible white vertices in $Q_1$, we conclude that the number of white vertices in $Q_1$ that are not in $N_3(v, w)$, and hence, the number of vertices in $X$ that belong to $Q_1$ is bounded by three. Similarly, the number of vertices in $X$ that belong to $Q_2$ is bounded by three. Moreover, the statement of the claim in **Case 1** carries in a straightforward manner to **Case 2**, and categories (iv) and (viii) contain at most one white vertex each. It follows that the number of white vertices in the set $X$, is bounded by eight. Now if $|X| = 8$, then both $Q_1$
and \( Q_2 \) (plus the black vertices in \( N_3(v, w) \) that reside in \( Q_1 \) and \( Q_2 \)) contain three white vertices. Since \( Q_1 \) contains exactly three white vertices, by Proposition 3.10, either \( Q_1^+ \neq \emptyset \), or \( Q_1 \) must contain an interior black vertex. If \( Q_1^+ \neq \emptyset \), since \( R \) is nice, the vertex in \( Q_1^+ \) must be connected to some vertex interior to \( Q_1 \) which must be black because the vertices in \( Q_1^+ \) are white. Therefore, if \( Q_1 \) contains exactly three white vertices, then there must exist an interior black vertex \( p \) in \( Q_1 \). Similarly, there must exist an interior black vertex \( q \) in \( Q_2 \). Since both \( p \) and \( q \) are black and are in \( N_2(v, w) \cup N_3(v, w) \), by Rule 4, \( p \) and \( q \) must dominate all vertices in \( N_3(v, w) \neq \emptyset \). In particular, \( p \) which is interior to \( Q_1 \) must dominate \( q \) which is interior to \( Q_2 \). This is a contradiction to the planarity of the graph. It follows that \( |X| \leq 7 \).

**Observation 2.** Every vertex in \( Y \) must dominate all vertices in \( N_3(v, w) \).

This observation follows from Rule 4 since the vertices in \( Y \) are black and are a subset of \( N_2(v, w) \cup N_3(v, w) \).

Let \( H \) be the graph obtained from \( G \) by identifying the vertex \( v \) with \( w \) along the path \((v, v_1, w, w_1, w)\). Clearly, \( H \) is planar. Let \( u \) be the resulting vertex by this identification. Let \( Y' \) be the set of vertices in \( Y \) that are in \( H \), and let \( y = |Y'| \). Similarly, let \( Z' \) be the set of vertices in \( Z \) that are in \( H \), and \( z = |Z'| \). Observe that the vertex \( u \) is connected to all the vertices in \( Y' \) and \( Z' \) in \( H \), and that the only vertices that might have been removed from \( H \) are boundary vertices to \( R \).

**Observation 3.** If \( y > 1 \) and \( z > 1 \), then the number of vertices in \( V(R) \) is bounded by 16.

Suppose that \( y > 1 \) and \( z > 1 \). If \( y > 2 \), since every vertex in \( Y' \) must dominate the vertices in \( Z' \), it follows that the subgraph of \( H \) induced by the set of vertices \( Y' \cup Z' \cup \{u\} \) contains a copy of \( K_{3,3} \), contradicting the planarity of \( H \) (The vertices in \( Y' \) form the first bipartition and the other vertices form the second bipartition). Suppose now that \( y = 2 \). If \( z > 2 \), then similarly, the subgraph induced by \( Z' \cup \{u\} \cup Y' \) contains a copy of \( K_{3,3} \) (the vertices in \( Y' \cup \{u\} \) form the first bipartition, and those in \( Z' \) form the second bipartition). Suppose now that \( y = z = 2 \). Then the number of vertices in \( X \cup Y' \cup Z' \) is bounded by 11. Since \( |V(R)| \leq |X \cup Y' \cup Z' \cup \{v_1, v_2, w_1, w_2\}| \), it follows that the number of vertices in \( V(R) \) is bounded by 16. 

Now we distinguish the following two subcases.

**Subcase 2.1.** \( z \leq 1 \). Let \( Y_1 = Y' \cap N_2(v, w) \), \( y_1 = |Y_1| \), \( Y_2 = Y - Y_1 \), and \( y_2 = |Y_2| \). Note that every vertex in \( Y \) must be connected to all vertices in \( Y_2 \cup Z \) by Rule 4. If \( y = y_1 + y_2 < 5 \), then since \( z \leq 1 \) and the number of vertices in \( X \) is bounded by 7 by Observation 1, the number of vertices in \( V(R) \) is bounded by 16. So we can assume that \( y \geq 5 \). If \( y_2 + z \geq 4 \), then the subgraph induced by the vertices \( \{u\} \cup Y_2 \cup Z \) is a copy of \( K_5 \). Thus, \( y_2 + z < 4 \). If \( y_2 + z = 3 \), then the subgraph induced by the bipartition \( (Y_2 \cup Z, Y_1 \cup \{u\}) \) contains a copy of \( K_{3,3} \), whereas if \( y_2 + z = 2 \), then subgraph induced by the bipartition \( (\{u\} \cup Y_2 \cup Z, Y_1) \) contains a copy of \( K_{3,3} \). Suppose now that \( y_2 + z = 1 \). If \( y_1 \leq 4 \), then \( y + z \leq 5 \), and hence, the number of vertices in \( R \) is bounded by 16. Assume now that \( y_1 \geq 5 \). Let \( p \) be the vertex in \( Y_2 \cup Z \), then \( p \) is connected to all vertices in \( Y_1 \) in \( H \), and hence in \( G \). Moreover, every vertex in \( Y_1 \) is either connected to \( v \) or \( w \) (or both) in \( G \). Since \( y_1 \geq 5 \), there must exist at least three vertices in \( Y_1 \) that are connected either to \( v \) or to \( w \) in \( G \). Let these vertices be \( p_1, p_2, \) and \( p_3 \), and assume without loss of generality, that these
vertices are connected to \( v \). Since \( p_1, p_2, \) and \( p_3 \), are also connected to \( p \), there must exist a vertex in \( \{p_1, p_2, p_3\} \), say \( p_2 \), that is interior to the region determined by \( v, p, \) and the other two vertices. But \( p_2 \in Y_1 \subseteq N_2(v, w) \), and hence \( p_2 \) must be connected to the boundary of \( R \). A contradiction. Thus, the number of vertices in \( V(R) \) is bounded by 16.

**Subcase 2.2.** \( y \leq 1 \). If \( z \leq 4 \), then \( y + z \leq 5 \), and given that \( |X| \leq 7 \) by Observation 1, the total number of vertices in \( V(R) \) is bounded by 16. Suppose that \( z \geq 5 \), and note that it must be the case that \( y \neq 0 \) because every vertex in \( N_3(v, w) \) must be connected to at least one vertex in \( N_2(v, w) \), and hence, if \( N_3(v, w) = \emptyset \), then \( N_2(v, w) = \emptyset \). Also note that all vertices in \( Z \) are connected to the single vertex \( q \) in \( Y \). Let \( p_1, p_2, p_3, p_4, \) and \( p_5 \) be vertices in \( Z \). All these vertices must be connected to \( q \), and to at least one vertex in \( \{v, w\} \). Thus, at least three vertices among \( \{p_1, p_2, p_3, p_4, p_5\} \) are connected either to \( \{v, q\} \), or to \( \{w, q\} \). Suppose, without loss of generality, that \( \{p_1, p_2, p_3\} \) are connected to \( v \) and \( q \). But then \( |W(v, q)| \geq 3 \), and it must be the case that at least one vertex in \( \{p_1, p_2, p_3\} \) is of degree 2, this contradicts Rule 8.

It follows that in all cases the number of vertices in \( V(R) \) is bounded by 16. This completes the proof.

**Theorem 3.12** The number of vertices in \( G \) is bounded by \( 67k \).

**Proof.** By Lemma 3.8, the number of category-(2) vertices in \( G \) is bounded by \( 18k \). Using this bound, we can assume that each region in \( \mathcal{R} \) is nice. By Corollary 3.4, the number of regions in \( \mathcal{R} \) is bounded by \( 3k - 6 \). According to Lemma 3.11, the number of vertices in \( V(R) \), where \( R \in \mathcal{R} \) is a nice region, is bounded by 16. It follows that the number of vertices in \( V(\mathcal{R}) \) is bounded by \( 48k - 96 \). Thus, the number of vertices in \( V(\mathcal{R}) \), and hence, in category (1), is bounded by \( 48k - 96 \) plus the number of vertices in \( D \) which are the endpoints of the regions in \( \mathcal{R} \). Therefore the number of vertices in \( V(\mathcal{R}) \) is bounded by \( 49k - 96 \), and the total number of vertices in \( G \) is bounded by \( 67k - 96 < 67k \). This completes the proof.

**Theorem 3.13** Let \( G \) be a planar graph with \( n \) vertices. Then in time \( O(n^3) \), computing a dominating set for \( G \) of size bounded by \( k \) can be reduced to computing a dominating set of size bounded by \( k \), for a planar graph \( G' \) of \( n' < n \) vertices, where \( n' \leq 67k \).

**Proof.** According to Theorem 2.1, in time \( O(n^3) \) we can construct a reduced graph \( G' \) from \( G \) where \( \gamma(G') = \gamma(G) \), and such that a dominating set for \( G \) can be constructed from a dominating set for \( G' \) in linear time. Moreover, the graph \( G' \) has no more than \( n \) vertices. If \( G \) has a dominating set of size bounded by \( k \), then \( G' \) has a dominating set of size bounded by \( k \) (since \( \gamma(G) = \gamma(G') \)), and by Theorem 3.13, we must have \( n' \leq 67k \). If this is the case, then we can work on computing a dominating set for \( G' \) of size bounded by \( k \), from which a dominating set for \( G \) can be easily computed. If this is not the case, then \( G \) does not have a dominating set of size bounded by \( k \), and the answer to the input instance is negative. This completes the proof.

### 4 A simple algorithm

In this section we present a simple algorithm for determining whether a graph \( G \) has a dominating set of size bounded by \( k \).
Let $G = (V, E)$ be a planar graph given with an embedding in the plane. The layer decomposition of $G$ with respect to the embedding, is a partitioning of $V$ into disjoint layers $(L_1, \ldots, L_r)$ defined inductively as follows. Layer $L_1$ is the set of vertices that lie on the outer face of $G$, and layer $L_i$ is the set of vertices that lie on the outer face of $G - \bigcup_{j=1}^{i-1} L_j$ for $1 < i \leq r$. It is well-known that a layer decomposition of a planar graph $G$ can be computed in linear time in the number of vertices in the graph [3].

A separator in a graph $G$ is a set of vertices $S$ whose removal disconnects $G$. If $(L_1, \ldots, L_r)$ is a layer decomposition of $G$, then clearly the vertices in any layer $L_i$ form a separator in $G$, separating the vertices in layers $L_1, \ldots, L_{i-1}$ from those in layers $L_{i+1}, \ldots, L_r$. Let $(G, k)$ be an instance of the planar dominating set problem. By Theorem 3.13, we can assume that $G$ is reduced, and that the number of vertices $n$ of $G$ satisfies $n \leq 67k$. Let $(L_1, \ldots, L_r)$ be a layer decomposition of $G$. Let $c > 0$ be a constant which will be determined later, and set $l = \lceil c \sqrt{k} \rceil$. Consider the families of layers $F_i, i = 1, \ldots, l$, where $F_i$ consists of layers $L_{1i}, L_{1i+1}, L_{1i+2}, \ldots$. Assume for now that the number of layers $r \geq l$. We will show later how to handle the situation when this is not the case. The families $F_i, i = 1, \ldots, l$, are disjoint, and each family forms a separator separating the graph into connected components that will be called chunks, where each chunk consists of at most $l$ consecutive layers. Since these $l$ families are disjoint and partition the layers into $l$ groups, and since the graph has at most $67k$ vertices, there exists an index $1 \leq \mu \leq l$, such that the number of vertices in $F_\mu$ is bounded by $67k/l$. Again, observe that the removal of $F_\mu$ from $G$ separates $G$ into chunks, each consisting of at most $l$ consecutive layers. Let these chunks be $G_1, \ldots, G_s$.

The basic idea behind the algorithm is to apply a simple divide-and-conquer strategy by removing the vertices in the family $F_\mu$ to split the graph into chunks, then to compute a minimum dominating set for the resulting chunks using the algorithm introduced in [17], which is a variation of Baker’s algorithm [4]. To do this, for each vertex $v$ in the $F_\mu$, we “guess” whether $v$ is in the minimum dominating set for $G$ or not (basically, what we mean by guessing is enumerating all sequences corresponding to the different possibilities). For each guess of all the vertices in $F_\mu$, we will solve the corresponding instance with respect to that guess. It was shown in [17] how this guessing process can be achieved using at most three statuses per vertex. Hence, guessing the vertices in $F_\mu$ can be done by enumerating at most $3|F_\mu| \leq 3^{67k/l}$ ternary sequences. After guessing each vertex in the separator and updating the graph accordingly, the instance becomes an instance of a variation of the minimum dominating set problem due to the constraints placed on some of the vertices in the graph. Kanj and Perković introduced an algorithm in [17], which is a variation of Baker’s algorithm [4], to solve this problem. The algorithm introduced in [17] solves this problem on the chunks in time $O(27^{d+1}n)$, where $d$ is the maximum number of layers in a chunk (i.e., the maximum depth of a chunk). Noting that $d \leq l$ and that $n \leq 67k$, we conclude that after guessing all the vertices in $F_\mu$, the problem can be solved in time $O(27^l k)$. If the number of layers $r$ in $G$ is less than $l$, we can simply call the algorithm in [17] directly on $G$ to solve the problem in time $O(27^l k)$. The algorithm is given in Figure 2 below.

It is not difficult to verify that the running time of the algorithm is $O(3^{67k/l} \cdot 27^l \cdot k + n^3)$, where the $O(n^3)$ time is to count for the time taken to reduce $G$ to its kernel. Niedermeier and Rossmanith showed how to get rid of the $k$ factor corresponding to the kernel size in the running time of such algorithms [19]. Using their techniques, the running time of the algorithm becomes $O(n^3 + 3^{67k/l} \cdot 27^l \cdot l)$. We choose $c$, and hence $l$, so that the above expressing is minimized. It can be shown that the expression is minimized when $c = \sqrt{67/3}$, and the running time of the algorithm becomes $O(n^3 + 245\sqrt{3})$.

**Theorem 4.1** In time $O(n^3 + 245\sqrt{3})$, it can be determined whether a graph on $n$ vertices has a
**Algorithm. DS-solver**

Input: a planar graph \( G \) of \( n \) vertices, and a parameter \( k \)

Output: a dominating set \( D \) of size \( \leq k \) in case it exists;

1. use the results in Theorem 3.13 to kernelize \( G \);
2. **if** the number of vertices \( n \) of \( G \) is \( > 67k \) **then**
   
   Stop("G does not have a dominating set of size \( \leq k \)"");
3. let \( c = \sqrt{67/3}; l = [c\sqrt{k}] \);
4. if the number of layers in \( G \) is \( < l \) **then**

   use the algorithm in [17] to solve the problem in time \( O(2^{45\sqrt{k}}) \); **Stop**;
5. let \( F_\mu \) be a separator of size \( \leq 67k/l \) separating the graph into chunks \( G_1, \ldots, G_s \) each consisting of at most \( l \) consecutive layers;
6. **for** each assignment to the vertices in \( F_\mu \), **do**
   
   update \( D \);
   
   split the graph into its components;
   
   compute a minimum dominating set \( D' \) for the resulting graph using the algorithm in [17];
   
   \( D = D \cup D' \);
7. output the smallest dominating set constructed in step 6 in case its size is bounded by \( k \); otherwise **return** ("G does not have a dominating set of size \( \leq k \)").

**Figure 2:** A simple algorithm solving planar-DS

**dominating set of size bounded by \( k \) or not.**

Theorem 4.1 shows that our algorithm for solving the planar-DS problem is competitive with the previous algorithms using the similar technique of layer decomposition of a planar graph [3, 17]. The above algorithm improves the \( O(2^{70\sqrt{k}}n) \) time algorithm given in [3] for the problem, and comes close to the \( O(2^{27\sqrt{k} + n}) \) time algorithm given in [17]. At the same time, our algorithm is much simpler than the algorithms in [3, 17], illustrating the power of kernelization in the process of designing efficient algorithms for parameterized NP-hard problems. Finally, we mention that recently Fomin and Thilikos presented an \( O(n^3 + 2^{15.13\sqrt{k}}) \) time algorithm to solve the planar-DS problem based on the concept of branch width [13].

**References**


