Algebra review

The following is a review of the algebra knowledge necessary to be successful in this course.

The topics covered include:
1. Grouping
2. Fractions
3. Exponents
4. Factoring
5. Inequalities
6. Logarithms

This material is taken from Appendix B on pages 509 – 518 in the textbook. More details and more examples are found there.

Grouping

Terms with a common symbol can be combined. We use the distributive laws to indicate how to do this.

Distributive laws:
\[ ac + bc = (a+b)c \]
\[ ac - bc = (a-b)c \]

In a bit, we will see how to use these laws directly. First, consider a more straightforward use of the equalities.

The distributive laws can also be used to simplify expressions by multiplying them out.

Examples:
\[ 2(x+1) = 2x + 2*1 = 2x + 2 \]
\[ 2(x+1) + 2(x-1) = 2x + 2 + 2x - 2 = 4x \]
Fractions

The following formulas are useful for adding, subtracting, and multiplying fractions:

**Rules for combining fractions:**

1. \[
a \frac{a}{c} + b \frac{c}{b} = \frac{a + b}{c}
\]
2. \[
a \frac{a}{c} - b \frac{c}{b} = \frac{a - b}{c}
\]
3. \[
a \frac{a}{c} + b \frac{d}{d} = \frac{ad + bc}{cd}
\]
4. \[
a \frac{a}{c} - b \frac{d}{d} = \frac{ad - bc}{cd}
\]
5. \[
a \frac{a}{c} \times b \frac{d}{d} = \frac{ab}{cd}
\]

**Examples:**

- \[
\frac{x-1}{2} + \frac{x+1}{2} = \frac{(x-1) + (x+1)}{2} = \frac{2x}{2} = x
\]
- \[
\frac{x-1}{2} \times \frac{x+1}{3} = \frac{3(x-1) - 2(x+1)}{6} = \frac{x-5}{6}
\]
- \[
\frac{2}{x} \times \frac{4}{y} = \frac{8}{xy}
\]

Exponents

If \( n \) is a positive integer and \( a \) is a real number, we define \( a^n \) as: \( a^n = a \times a \times \ldots \times a \) where there are \( n \) repetitions of the \( a \) term.

There are some special cases to consider:

1. If \( a \) is a nonzero, real number we define \( a^0 = 1 \).
2. If \( n \) is a negative integer, and \( a \) is a nonzero real number, then \( a^n = \frac{1}{a^{-n}} \)

**Examples:**

- \( 2^4 = 2 \times 2 \times 2 \times 2 = 16 \)
- \( 2^{-4} = \frac{1}{2^4} = \frac{1}{16} \)
If \( a \) is a positive, real number, the definition of \( a^x \) can be extended to include all real numbers \( x \) (rational or irrational).

**Laws of exponents**: Let \( a \) and \( b \) be positive real numbers, and let \( x \) and \( y \) be real numbers. Then:

1. \( a^{x+y} = a^x a^y \)
2. \( (a^x)^y = a^{xy} \)
3. \( \frac{a^x}{a^y} = a^{x-y} \)
4. \( a^x b^x = (ab)^x \)
5. \( \frac{a^x}{b^x} = \left( \frac{a}{b} \right)^x \)

**Examples**:
- \( 3^{2+4} = 3^2 \times 3^4 = 9 \times 81 = 729 = 3^6 \)
- \( (3^2)^3 = 3^8 = 6561 = 9^4 \)
- \( 3^2 \times 4^2 = (3 \times 4)^2 = 12^2 = 144 = 9 \times 16 \)
- \( 2^x \times 2^x = 2^{2x} = 4^x \)

**Factoring**

We can use the equation

\[(x + b)(x + d) = x^2 + (b+d)x + bd\]

to factor an expression of the form \( x^2 + c_1 x + c_2 \).

A generalization of the equation above has the following form:

\[(ax+b)(cx+d) = (ac)x^2 + (ad +bc)x + bd\]

that can be used to factor expressions of the form \( d_1 x^2 + d_2 x + d_3 \).

There are also three special cases to consider:

1. \( (x+b)^2 = x^2 + 2bx + b^2 \)
2. \( (x-b)^2 = x^2 - 2bx + b^2 \)
3. \( (x+b)(x-b) = x^2 - b^2 \)
Example 1

Factor the expression $x^2 + 3x + 2$.

By the equation above, we have that $b+d = 3$ and $bd = 2$.

If $bd = 2$, and $b$ and $d$ are integers, then the only choices for $b$ and $d$ are 1, 2, and -1, -2.

Since $b+d = 3$, we can rule out the negative values, so that $b$ and $d$ must be 1 and 2.

Thus $x^2 + 3x + 2 = (x + 1)(x + 2)$.

Example 2

Factor the expression $x^2 - 36$.

Since $36 = 6^2$, we have special case 3 here.

Thus $x^2 - 36 = (x+6)(x-6)$.

Example 3

Factor the expression $6x^2 - x - 2$.

This has the form of the generalized equation above, so that

$$ac = 6, \ ad + bc = -1, \ and \ bd = -2$$

Since $ac = 6$, there are four possibilities for $a$ and $c$: 1,6; 2,3; -1,-6; and -2, -3.

Since $bd = -2$, the only possibilities for $b$ and $d$ are: 1, -2; -1, 2.

Since $ad + bc = -1$, it must be that $a = 2$, $b = 1$, $c = 3$, and $d = -2$.

Thus $6x^2 - x - 2 = (2x + 1)(3x - 2)$. 
Example 4

Show that \[
\left(\frac{n(n+1)}{2}\right)^2 + (n + 1)^2 = \frac{(n + 1)^2(n^2 + 4)}{4}
\]

\[
\left(\frac{n(n+1)}{2}\right)^2 + (n + 1)^2 = \frac{n^2(n+1)^2}{2^2} + (n + 1)^2
\]

\[
= \frac{n^2(n+1)^2}{4} + \frac{4(n + 1)^2}{4}
\]

\[
= \frac{n^2(n+1)^2 + 4(n + 1)^2}{4}
\]

\[
= \frac{(n+1)^2(n^2 + 4)}{4}
\]

Example 5

Show that \[
\left(\frac{n(n+1)}{2}\right)^3 + (n + 1)^2 = \frac{(n + 1)^2[n^4 + n^3 + 8]}{8}
\]

\[
\left(\frac{n(n+1)}{2}\right)^3 + (n + 1)^2 = \frac{n^3(n+1)^3}{2^3} + (n+1)^2
\]

\[
= \frac{n^3(n+1)^3}{8} + \frac{8(n + 1)^2}{8}
\]

\[
= \frac{n^3(n+1)^3 + 8(n + 1)^2}{8}
\]

\[
= \frac{(n+1)^2 + [n^3(n+1) + 8]}{8}
\]

\[
= \frac{(n+1)^2[n^4 + n^3 + 8]}{8}
\]

**Inequalities**

The following rules about inequalities can be helpful when we manipulate asymptotic notation later in the course.
Laws of Inequalities:
1. If \( a < b \) and \( c \) is any number then \( a+c < b+c \).
2. If \( a \leq b \) and \( c \) is any number then \( a+c \leq b+c \).
3. If \( a > b \) and \( c \) is any number then \( a+c > b+c \).
4. If \( a \geq b \) and \( c \) is any number then \( a+c \geq b+c \).
5. If \( a < b \) and \( c > 0 \) then \( ac < bc \).
6. If \( a \leq b \) and \( c > 0 \) then \( ac \leq bc \).
7. If \( a < b \) and \( c < 0 \) then \( ac > bc \).
8. If \( a \leq b \) and \( c < 0 \) then \( ac \geq bc \).
9. If \( a > b \) and \( c > 0 \) then \( ac > bc \).
10. If \( a \geq b \) and \( c > 0 \) then \( ac \geq bc \).
11. If \( a > b \) and \( c < 0 \) then \( ac < bc \).
12. If \( a \geq b \) and \( c < 0 \) then \( ac \leq bc \).
13. If \( a < b \) and \( b < c \) then \( a < c \).
14. If \( a < b \) and \( b \leq c \) then \( a < c \).
15. If \( a \leq b \) and \( b < c \) then \( a < c \).
16. If \( a \leq b \) and \( b \leq c \) then \( a \leq c \).
17. If \( a > b \) and \( b > c \) then \( a > c \).
18. If \( a > b \) and \( b \geq c \) then \( a > c \).
19. If \( a \geq b \) and \( b > c \) then \( a > c \).
20. If \( a \geq b \) and \( b \geq c \) then \( a \geq c \).

Keep this list of inequalities and use it when you are doing the homework.

Consider some examples of how to apply these laws:

**Example 1**: Solve the inequality \( x - 5 < 6 \).

We can add 5 to both sides and preserve the inequality.
Thus \( x - 5 + 5 < 6 + 5 \) so that \( x < 11 \).

**Example 2**: Solve the inequality \( 3x + 4 < x + 10 \).

We can add \(-x\) to both sides and preserve the inequality.
Thus \( 3x + 4 - x < x - x + 10 \). This gives \( 2x + 4 < 10 \).
Adding \(-4\) to both sides we get \( 2x + 4 - 4 < 10 - 4 \).
This yields \( 2x < 6 \). Multiplying both sides by \( \frac{1}{2} \), which is a positive value, we get \( x < 3 \).

**Example 3**: Show that if \( n > 2m \) and \( m > 2p \) then \( n > 4p \).

Since 2 is positive we can multiply both sides of \( m > 2p \) to obtain \( 2m > 4p \).
Since $n > 2m$, by one of the above rules we know that $n > 4p$.

**Logarithms**

Assume that $b$ is a positive real number not equal to 1.

If $x$ is a positive real number, the **logarithm to the base** $b$ of $x$ is the exponent to which $b$ must be raised to obtain $x$.

We denote this by $\log_b x$.

If $y = \log_b x$ then $b^y = x$ by the definition of the logarithm.

**Example 1**: \( \log_2 8 = 3 \) since \( 2^3 = 8 \).

**Notation**: \( \log_2 x \) is denoted \( \lg x \).

The following are useful facts concerning logarithms.

**Laws of Logarithms** : Suppose that $b > 0$ and $b \neq 1$.

1. \( b^{\log_b x} = x \)
2. \( \log_b (xy) = \log_b x + \log_b y \)
3. \( \log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y \)
4. \( \log_b (x^y) = y \log_b x \)
5. \( \log_b b^y = x \) (follows from #1 and #4)
6. If $a > 0$ and $a \neq 1$, we have \( \log_a x = \frac{\log_b x}{\log_b a} \)
7. If $x > y > 0$, then $\log_b x > \log_b y$.

Note: The sixth fact is called the **change-of-base formula** for logarithms. It allows us to switch bases for logarithms.

**Example 2**: \( \lg(x/4) = \lg x - \lg 4 = \lg x - 2 \).

**Example 3**: \( \lg((16x)^4) = 4(\lg 16 + \lg x) = 4(\lg x + 4) \)

**Example 4**: \( \lg(16x^4) = \lg 16 + \lg x^4 = 4 + 4\lg x = 4(\lg x + 1) \)

**Example 5**: Show that if $k$ and $n$ are positive integers satisfying \( 2^{k-1} < n < 2^k \) then $k-1 < \log n < k$.

By law #7, the logarithm function is increasing, so that:
\[ \log(2^{k-1}) < \log n < \log(2^k) \]

By law #5, \( \log(2^{k-1}) = k-1 \) and \( \log(2^k) = k \).

Thus \( k-1 < \log n < k \) as required.

**Practice problems**

If you feel you need more practice, please try the following exercises.

1. Exercises 1 and 3, page 518.
2. Exercises 4, 5 and 7, page 518.
3. Exercises 8 and 11, page 518.
5. Exercises 16, 19, and 22, page 518.
8. Exercises 54 and 57, page 519.

Note that exercises with red numbering have partial or full solutions listed in the back of the book.